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A Generalised Coprime Graph-Revisited

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Abstract

In this paper we consider the Generalised coprime graph which is denoted by $G(n, M)$, whose vertex set is $\{1, 2, \dots, n\}$ and an edge, say ab, of $G(n, M)$ is defined when $\gcd(a, b) \in M \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$. In this article we obtain new expressions for the degree sequence of $G(n, \{1\})$ and the clique number for $G(n, \{1\})$. Further, equivalent conditions for connectivity of $G(n, M)$, existence of isolated vertex in $G(n, M)$ and bipartiteness of $G(n, M)$ were obtained.

Key words: Coprime graph, Clique number, Degree

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1 Motivation

The coprime graph of integers is a simple undirected graph with the vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{ab : a, b \in V \text{ and } \gcd(a, b) = 1\}$. Paul Erdös and Gabor N. Sarkozy [7] addressed an extremal graph theory problem over this graph. They have obtained the maximal size of induced subgraph that guarantees the existence of a cycle in it of a specific odd length. In another paper, Gabor N. Sarkozy [2] arrived at a lower bound on the size of induced subgraph that guarantees the existence of a special type of tripartite graph.

A generalisation of coprime graph was given by Mutharasu et al., see [6]. In this article we explore some new properties of this generalised graph.

Following is the definition of generalised coprime graph given by Mutharasu et al. [6].

Definition 1.1. Let n be a positive integer and let $M \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. A graph, denoted $G(n, M)$, is defined with vertex set $\{1, 2, \ldots, n\}$ and an edge ab of $G(n, M)$ is defined when $\gcd(a, b) \in M$. We call this graph as *gcd-graph*.

The following graph illustrates this definition:

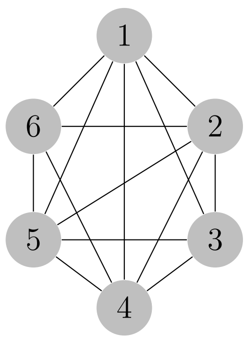

Figure 1: $G(6, \{1, 2\})$

This article is organised as follows: in Section 2 we find new expression for the degree sequence of $G(n, \{1\})$ and bounds for the maximum and minimum degree of $G(n, M)$. In section 3, we obtain conditions over M that makes $G(n, M)$: (i) a connected graph, and (ii) a graph with isolated vertex. Consequently, the number of connected gcd-graphs were counted. As the further development of coprime graph study, we find the clique number of $G(n, \{1\})$ and the Dirichlet degree sum: $\sum_{v|n} \deg(v)$ of $G(n, \{1\})$. A necessary condition for $G(n, M)$ to be a bipartite graph is also obtained.

Throughout this paper we use the graph terminologies of Gary Chartrand and Ping Zhang [3].

2 Degree sequence of $G(n, M)$

The first part of this section is concerned with the study of degree sequence of the graph $G(n, \{1\})$ which is the actual coprime graph considered by Paul Erdös.

Notation 2.1. We denote the degree of vertex v of the graph $G(n, \{1\})$ by $\deg_v(n)$.

The following result obtained by Junyao Pan and Xiuyun Guo[4] puts $deg_v(n)$ as an expression of inclusion-exclusion type.

Theorem 2.2. Let n be a positive integer and let $v \in \{2, \ldots, n\}$. Let $v = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the prime factorisation of v. Then $\deg_v(n) = n - \sum_{p_i} \lfloor \frac{n}{p_i} \rfloor + \sum_{p_i < p_j} \lfloor \frac{n}{p_i p_j} \rfloor - \ldots$

As a consequence of Theorem 2.2 we obtain that the Dirichlet sum $\sum_{v|n} \deg_v(n)$ is one less than the gcd-sum function.

Theorem 2.3. In $G(n, \{1\})$, we have $\sum_{v|n} \deg_v(n) = g(n) - 1$, where $g(n)$ denotes the gcd-sum function defined by $g(n) = \sum_{k=1}^{n} \gcd(n, k)$.

Proof: Let $v \geq 2$ be a divisor of n. Then from Theorem 2.2 we have

$$\begin{aligned}\deg_v(n) &= n - \sum_{p_i} \frac{n}{p_i} + \sum_{p_i < p_j} \frac{n}{p_i p_j} - \ldots \\ &= \frac{n}{v} \left(v - \sum_{p_i} \frac{v}{p_i} + \sum_{p_i < p_j} \frac{v}{p_i p_j} - \ldots \right) \\ &= \frac{n}{v} \phi(v),\end{aligned}$$

where $\phi(n)$ is the Euler's phi function that counts the number of positive integers that are less than n and relatively prime to n.

For $v = 1$, we have

$$\begin{aligned}\deg_v(n) &= n - 1 \\ &= \frac{n}{1} \phi(1) - 1.\end{aligned}$$

This gives the relation: $\sum_{v|n} \deg_v(n) = \sum_{v|n} \frac{n}{v} \phi(v) - 1$.

Kevin A. Broughan [5] established that $g(n) = \sum_{k=1}^{n} \gcd(n, k) = \sum_{v|n} \frac{n}{v} \phi(v)$.

Now the result follows from the two equations above. ■

In the proof above, we have the expression: $\deg_v(n) = \frac{n}{v} \phi(v)$ when $v \geq 2$ is a divisor of n. A general formula of this fashion is obtained in the following result. To present that result we need the following definition.

Definition 2.4. Let n and m be two positive integer. We define $\phi(n,m)$ to be the number of positive integers that are less than or equal to m and relatively prime to m.

Theorem 2.5. Let n and v be two positive integers with $v \geq 2$. We have $\deg_v(n) = k\phi(v) + \phi(v,r)$, where k (resp. r) is the quotient (resp. remainder) while dividing n by v.

Proof: Let $v \in \{1,2,\ldots,n\}$. Consider the equality $n = kv + r$. From the relation

$$\gcd(v,i) = \gcd(v, v+i),$$

it follows that the number of integers in $\{1,2,\ldots,n\}$ that are relatively prime to v equals $k\phi(v) + \phi(v,r)$. Since this counting is $\deg_v(n)$, the result follows. ∎

In the following result, we have a recurrence type expression for $\deg_v(n)$.

Theorem 2.6. Let n and v be two positive integers with $1 \leq v \leq n$. Then we have

$$\deg_v(n+1) = \begin{cases} \deg_v(n) & \text{if } \gcd(v,n+1) \neq 1; \\ \deg_v(n)+1 & \text{if } \gcd(v,n+1) = 1. \end{cases}$$

Proof: Suppose that $\gcd(v,n+1) = 1$. Then exactly an edge $(v,n+1)$ is incident with v in $G(n+1,\{1\})$ other than the edges incident with v in $G(n,\{1\})$. Therefore we have $\deg_v(n+1) = \deg_v(n) + 1$.

Suppose that $\gcd(v,n+1) \neq 1$. Then evidently $n+1$ is not adjacent with v. Therefore the number of edges that are incident with v in $G(n,\{1\})$ remains the same in $G(n+1,\{1\})$. Hence $\deg_v(n+1) = \deg_v(n)$. ∎

It is evident that $G(n, \{1,2,\cdots,\lfloor \frac{n}{2} \rfloor\})$ is a complete graph of order n. This fact is used to arrive at the following identity.

Theorem 2.7. We have $\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \Big(\phi(1) + \phi(2) + \ldots + \phi(\lfloor \frac{n}{k} \rfloor) - 1\Big) = \frac{n(n-1)}{2}$.

Proof: Consider the graph $G(n,\{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\})$. Since $1 \leq \gcd(a,b) \leq \lfloor \frac{n}{2} \rfloor$ for every $a,b \in \{1,2,\ldots,n\}$, it follows that $G(n,\{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\})$ is a complete graph. Then it is evident that the number of edges in $G(n,\{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\})$ is $\frac{n(n-1)}{2}$.

We count the number of edges in $G(n,\{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\})$ in a different way. Consider the following array of lattice points:

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & \cdots & (1,n-1) & (1,n) \\ \cdot & (2,2) & (2,3) & \cdots & (2,n-1) & (2,n) \\ \cdot & \cdot & (3,3) & \cdots & (3,n-1) & (3,n) \\ \vdots & \vdots & \vdots & \ddots & (n-1,n-1) & (n-1,n) \\ \cdot & \cdot & \cdot & \cdot & \cdot & (n,n) \end{bmatrix}.$$

we see that the number of lattice points (a, b) in the r th column that satisfies the condition $\gcd(a, b) = 1$ is $\phi(r)$. Consequently, the number of lattice points (a, b) in the above array such that $\gcd(a, b) = 1$ is $\phi(1)+\phi(2)+\cdots+\phi(n)$. In the same way, for $k > 1$, the number of lattice points (a, b) in the kr th column that satisfies the condition $\gcd(a, b) = k$ is $\phi(r)$. This gives the conclusion that the number of lattice points (a, b) satisfying the condition $\gcd(a, b) = k$ in the above array equals $\phi(1) + \phi(2) + \ldots + \phi(\lfloor \frac{n}{k} \rfloor)$. At this juncture, we observe that $\gcd(a, b)$ varies from 1 to $\lfloor \frac{n}{2} \rfloor$. Now we see that except the lattice points of the form (t, t), all the other lattice points of the above array may serve as the edges of a complete graph with n vertices. Therefore, the number of edges (a, b) in $G(n, \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\})$ satisfying $\gcd(a, b) = k$ equals $\phi(1) + \phi(2) + \ldots + \phi(\lfloor \frac{n}{k} \rfloor) - 1$. Since k varies from 1 to $\lfloor \frac{n}{2} \rfloor$, the result follows. ■

Next result gives bounds and expression for the maximum and minimum degree of $G(n, M)$.

Theorem 2.8. Let $\Delta(G(n, M))$ (resp. $\delta(G(n, M))$) denotes the maximum (resp. minimum) degree of the graph $G(n, M)$. Then we have

(a) $\Delta(G(n, M)) \geq \min_{m\in M} \lfloor \frac{n}{m} \rfloor - 1$ if $1 \notin M$

(b) $\Delta(G(n, M)) = n - 1$ if $1 \in M$

(c) $\delta(G(n, M)) = 0$ if $1 \notin M$

(d) let $\delta(n)$ denote the minimum degree of $G(n, \{1\})$ and let $M_n = \{v : deg_v(n) = \delta(n)\}$. We have

$$\delta(n+1) = \begin{cases} \delta(n) & \text{if } \gcd(v, n+1) \neq 1 \text{ for each } v \in M_n \text{ and } \phi(n+1) \geq \delta(n) + 1; \\ \delta(n) + 1 & \text{if } \gcd(v, n+1) = 1 \text{ for some } v \in M_n \text{ and } \phi(n+1) \geq \delta(n) + 1; \\ \phi(n+1) & \text{if } \phi(n+1) < \delta(n) + 1. \end{cases}$$

Proof: Assume $1 \notin M$. We see that for each $m \in M$, the induced subgraph by the vertices $m, 2m, 3m, \ldots$ is isomorphic to $G(\lfloor \frac{n}{m} \rfloor, \{1\})$. Since $\Delta(G(\lfloor \frac{n}{m} \rfloor, \{1\}) = \lfloor \frac{n}{m} \rfloor - 1$ and the vertex

m assumes the maximum degree in the induced subgraph by the vertices $m, 2m, 3m, \ldots$, it follows that $\Delta(G(n,M)) \geq \min_{m\in M}\lfloor\frac{n}{m}\rfloor - 1$. Now (a) follows.

Suppose that $1 \in M$. Then it follows that the vertex 1 is adjacent with all the other vertices, whence we have $\Delta(G(n,M)) = n-1$. Now (b) follows.

Assume that $1 \notin M$. Then the vertex 1 will be isolated one. This gives $\delta(G(n,M)) = 0$. Now (c) follows.

Add the vertex $n+1$ with $G(n,\{1\})$ and join an edge with each vertex, say v, of $G(n,\{1\})$ if $\gcd(v,n+1) = 1$. Note that the resulting graph is $G(n+1,\{1\})$. Also the introduction of $n+1$ implies that $\delta(n+1) = \delta(n)$ or $\delta(n+1) = \delta(n)+1$ provided $\deg(n+1) \geq \delta(n)+1$. More precisely, when $\deg(n+1) \geq \delta(n)+1$, one can see that $\delta(n+1) = \delta(n)$ if $\gcd(v,n+1) \neq 1$ for each $v \in M_n$, and $\delta(n+1) = \delta(n)+1$ if $\gcd(v,n+1) = 1$ for some $v \in M_n$. On the otherhand, if $\deg(n+1) < \delta(n)+1$, then $\delta(n+1) = \deg(n+1)$. Since $\deg(n+1) = \phi(n+1)$, part (c) follows. ∎

3 Complement, Connectedness and Clique number of $G(n, M)$

Recall from the Definition 1.1 that the graph $G(n,M)$ is called gcd-graph. Recently, Ethan Berkove and Michael Brilleslyper [1] studied the complement of a generalised coprime graph in various aspects. In the following result, we will show that complement of a gcd-graph is again a gcd-graph.

Theorem 3.1. We have $\overline{G(n,M)} = G(n,\mathbb{N}\setminus M)$.

Proof: Let ab be an edge in $\overline{G(n,M)}$. Then we have $\gcd(a,b) \notin M$. That is, $\gcd(a,b) \in \mathbb{N}\setminus M$. In other sense, the edge ab lies in $G(n,\mathbb{N}\setminus M)$. Consequently, $\overline{G(n,M)} \subseteq G(n,\mathbb{N}\setminus M)$. Let uv be an edge in $G(n,\mathbb{N}\setminus M)$. Then we have $\gcd(u,v) \notin M$. That is, uv is not an edge of $G(n,M)$. Equivalently, uv is an edge of $\overline{G(n,M)}$. Therefore $G(n,\mathbb{N}\setminus M) \subseteq \overline{G(n,M)}$. Consequently, $\overline{G(n,M)} = G(n,\mathbb{N}\setminus M)$. ∎

Now we find equivalent condition for connectivity of $G(n,M)$.

Theorem 3.2. A gcd graph $G(n,M)$ is connected if, and only if, $1 \in M$.

Proof: Assume that $G(n,M)$ is connected. If $1 \notin M$, then the vertex 1 will be an isolated vertex, which is not the case. Therefore $1 \in M$. To prove the converse, assume that $1 \in M$. Then it is easy to see that the vertex 1 is adjacent with every other vertices of $G(n,M)$. Now the connectivity of $G(n,M)$ is assured. ∎

The above characterisation for connectedness in $G(n, M)$ permits us to count the number of connected gcd graphs.

Theorem 3.3. We have

(a) The number of connected labelled gcd graphs with n vertices is $2^{\lfloor \frac{n}{2} \rfloor - 1}$.

(b) The number of non connected labelled gcd graphs with n vertices is $2^{\lfloor \frac{n}{2} \rfloor - 1}$.

Proof: In view of Theorem 3.2 one can map a connected labelled gcd graph on n vertices with a subset of $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ having 1, and vice versa. Here we take a subset of $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$, because every integer greater than $\lfloor \frac{n}{2} \rfloor$ will never contribute to $\gcd(a, b)$ for $a, b \in \{1, \ldots, n\}$. Since the number of subsets of afore mentioned type is $2^{\lfloor \frac{n}{2} \rfloor - 1}$, part (a) follows.

In similar way, in view of Theorem 3.2 one can map a non-connected labelled gcd graph on n vertices with a subset of $\{2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and vice versa. Since the number of subsets of this type is $2^{\lfloor \frac{n}{2} \rfloor - 1}$, part (b) follows. ∎

Now we have another consequence of Theorem 3.2 .

Theorem 3.4. Every gcd graph is non self-complementary.

Proof: Assume that $G(n, M)$ is self-complementary.

Case i. Assume that $G(n, M)$ is connected. Then from Theorem 3.2 it follows that $1 \in M$. Since $\overline{G(n, M)} = G(n, \overline{M})$, from Theorem 3.2 it follows that $\overline{G(n, M)}$ is disconnected. This contradict our assumption.

Case ii. Assume that $G(n, M)$ is disconnected. Then from Theorem 3.2 it follows that $1 \notin M$. Since $\overline{G(n, M)} = G(n, \overline{M})$, from Theorem 3.2 it follows that $\overline{G(n, M)}$ is connected. This contradict our assumption. ∎

Remark 3.5. There are many other simple consequences of Theorem 3.2. We list few of them.

1. For $n \geq 3$, $G(n, M)$ is not a tree.

2. For $n \geq 3$, $G(n, M)$ is not a cycle.

3. Since every k-regular graph of order n with $k \geq \lfloor \frac{n}{2} \rfloor$ is a Hamiltonian graph and complement of a k-regular graph is $n - 1 - k$ regular, in view of Theorem 3.2, $G(n, M)$ is always a non-regular graph.

Now we have a characterisation theorem for the existence of isolated vertex in $G(n, M)$.

Theorem 3.6. A vertex v in $G(n, M)$ is isolated if, and only if, none of the divisors of v is in M.

Proof: Assume that v is an isolated vertex of $G(n, M)$. Now in view of Theorem 3.2, we have that $1 \notin M$. Again since v is isolated, we have that $\gcd(v, k) \notin M$ for every $k \in \{1, \ldots, n\}$. If a divisor of v, say d with $d > 1$, is in M, then $\gcd(v, d) = d$. Consequently, there is an edge, say vd, in $G(n, M)$, but this is not the case.

Conversely, assume that none of the divisors of v is in M for some v. We show that v is an isolated vertex. If not, then we have $\gcd(v, k) \in M$ for some $k \in \{1, \ldots, n\}$. This show that a divisor of v is present in M, which is not the case. ∎

Next result is a sufficient condition for the existence of a spanning complete graph of order m (denoted by K_m) in $G(n, M)$.

Theorem 3.7. If $\{a, 2a \cdots, \lfloor \frac{m}{2} \rfloor a\} \subseteq M$ and $ma \leq n$, then $G(n, M)$ has K_m as an induced subgraph.

Proof: For each $k_1, k_2 \in \{a, 2a, \ldots, ma\}$ it is evident to see that $\gcd(k_1, k_2) \in \{a, 2a, \ldots, \lfloor \frac{m}{2} \rfloor a\}$. Whence it follows that the vertices $\{a, 2a, \ldots, ma\}$ forms a spanning subgraph K_m. ∎

The converse of the theorem above cannot be asserted. Following graph serves as an example for this:

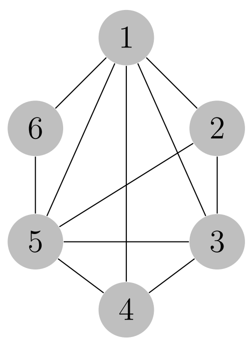

Figure 2: $G(6, \{1\})$

Here the vertices $\{1, 3, 4, 5\}$ induces a K_4, but the set $\{1\}$ does not meet the hypothesis of the above theorem. In fact one can even assert a more strong statement than this, that is, for any given positive integer m one can find an n such that the graph $G(n, \{1\})$ has K_m as an induced subgraph.

Theorem 3.8. Let $n \geq 2$ be a positive integer. Then $G(n, \{1\})$ has $K_{\pi(n)+1}$ as an induced subgraph, where $\pi(n)$ denotes the number of primes less than or equal to n.

Proof: Consider the set $\{1\} \cup \{p \in \mathbb{N} : p \text{ is a prime and } p \leq n\}$. Then it is easy to see that the cardinality of this set is $\pi(n) + 1$ and that every two elements of this set are relatively prime. From this it follows that the vertices $\{1\} \cup \{p \in \mathbb{N} : p \text{ is a prime and } p \leq n\}$ induces a complete graph of order $\pi(n) + 1$. Now the result follows. ∎

Next result asserts that the maximal order of an induced complete subgraph of $G(n, \{1\})$ is $\pi(n) + 1$.

Theorem 3.9. The clique number of $G(n, \{1\})$ is $\pi(n) + 1$.

Proof: Let $M(n)$ be the cardinality of maximal subset, say $M_{\max}$, of $\{1, 2, \ldots, n\}$ such that $a, b \in M_{max}$ imply that $\gcd(a, b) = 1$. Now we claim that $M(n) = \pi(n)+1$. It is evident to see that every pair of distinct elements in the set $\{1\}\cup\{p \in \mathbb{N} : p \text{ is a prime and } p \leq n\}$ is relatively prime. Hence, we have $M(n) \geq \pi(n) + 1$. Suppose that there is a set of integers, say K, with every distinct pair of elements of K being relatively prime and $|K| > \pi(n)+1$. If either 1 or one of the primes less than n is not present in K, then from the cardinality of K and the fundamental theorem of algebra it follows that there will be atleast one pair of distinct elements of K which is not relatively prime. Consequently, $\{1\} \cup \{p \in \mathbb{N} : p \text{ is a prime and } p \leq n\} \subseteq K$. If $K = \{1\} \cup \{p \in \mathbb{N} : p \text{ is a prime and } p \leq n\}$ then we are done, or else again from the cardinality of K and the fundamental theorem of algebra it follows that there will be atleast one pair of distinct elements of K which is not relatively prime. Consequently, $M(n) = \pi(n) + 1$. ∎

The final result of this section is a necessary condition for $G(n, M)$ to be bipartite.

Theorem 3.10. If $G(n, M)$ is bipartite then $m > \frac{n}{3}$ for every $m \in M$.

Proof: We prove the contrapositive of the above statement. Suppose that $m \leq \frac{n}{3}$ for some $m \in M$. Now since $\gcd(m, 2m) = m$, $\gcd(2m, 3m) = m$ and $\gcd(m, 3m) = m$, we have the triangle $m - 2m - 3m - m$. Since every bipartite graph must necessarily be void of odd cycle, the result follows. ∎

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