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Bounds for the spectral radius of divisor degree matrix and divisor degree energy of graphs

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Abstract

Gutman was the first to introduce the energy of a graph G. With this motivation we newly defined a matrix called divisor degree matrix and from that we obtained divisor degree energy of a simple graph G. In this paper, we obtain the bounds for the spectral radius γ_1 of divisor degree matrix for graph G. Also, we obtain the bounds for the divisor degree energy of graph G.

Key words: Energy, Divisor degree matrix, Divisor degree energy, spectral radius

2010 Mathematics Subject Classification : 05C50

1 Introduction

There are different types of matrices that are associated with graphs and its energies are studied in [5, 7, 8, 18, 19, 20]. Let G be a simple graph with n vertices $v_1, v_2, ..., v_n$ and m edges. If a vertex v_i is adjacent to v_k, then we write it as $v_i v_k \epsilon E(G)$. Let d_k be a degree of a vertex v_k, $k = 1, 2, ..., n$ with maximum degree Δ and minimum degree δ respectively. The adjacency matrix $A(G)$ of a graph G is a real symmetric matrix with n vertices is defined as $a_{ik} = 1$ if $v_i v_k \epsilon E(G)$ and zero otherwise. Then the $n \times n$ matrix has its eigenvalues in non-increasing

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order $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, where λ_1 is the greatest eigenvalue of $A(G)$. Gutman [9] was the first to introduce the energy of a graph G in 1978 as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

With this motivation on energy of a graph, we introduced a new matrix of a graph called divisor degree matrix [13]. The *divisor degree matrix* $\mathfrak{D}\mathfrak{D}(G)$ of a graph G is a real symmetric matrix with n vertices is defined as

$$dd_{ik} = \begin{cases} \left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right] & \text{if } v_i and v_k are\ adjacent\ and\ d_i \neq d_k \\ 1 & \text{if } v_i \text{ and } v_k \text{ are adjacent and } d_i = d_k \\ 0 & \text{ otherwise} \end{cases}$$

where $[x]$ denotes an integral part of real number x. Then the $n \times n$ real symmetric matrix has its eigenvalues in non-increasing order as $\gamma_1 \geq \gamma_2 \geq ... \geq \gamma_n$, where γ_1 is the spectral radius of divisor degree matrix of G. The *divisor degree energy* (DDE) is defined as

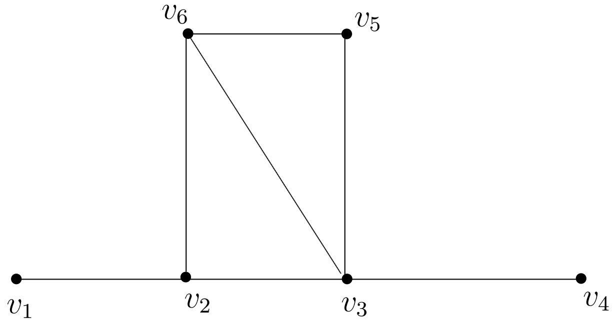
$$E_{\mathfrak{D}\mathfrak{D}}(G) = \sum_{i=1}^{n} |\gamma_i| . \tag{1}$$

From this, we observed that the adjacency matrix and divisor degree matrix of a regular graph are the same. So, we have the following results in [1, 9] as follows:

$(i)\mathrm{E}_{\mathfrak{D}\mathfrak{D}}(K_n) \ = 2(n-1).$

$$(ii)\mathrm{E}_{\mathfrak{D}\mathfrak{D}}(C_n) = \begin{cases} 2\csc \frac{\pi}{2n}, & \text{if } n \equiv 1(mod\ 2) \\ 4\csc \frac{\pi}{n}, & \text{if } n \equiv 2(mod\ 4) \\ 4\cot \frac{\pi}{n}, & \text{if } n \equiv 0(mod\ 4) \end{cases}$$

Example 1.1. Consider the graph G.



The divisor degree matrix of the graph G is

$$\mathfrak{DD}(G)=\begin{bmatrix}0&3&0&0&0&0\\3&0&1&0&0&1\\0&1&0&4&2&1\\0&0&4&0&0&0\\0&0&2&0&0&1\\0&1&1&0&1&0\end{bmatrix}$$

The characteristic polynomial of the divisor degree matrix $\mathfrak{DD}(G)$ is

$$|\gamma I-\mathfrak{DD}(G)|=\begin{vmatrix}\gamma&-3&0&0&0&0\\-3&\gamma&-1&0&0&-1\\0&-1&\gamma&-4&-2&-1\\0&0&-4&\gamma&0&0\\0&0&-2&0&\gamma&-1\\0&-1&-1&0&-1&\gamma\end{vmatrix}$$

The characteristic polynomial is $\gamma^6-33\gamma^4-6\gamma^3+231\gamma^2+36\gamma-144$ and the divisor degree eigenvalues of G are $\gamma_1=4.985, \gamma_2=2.923, \gamma_3=-4.664, \gamma_4=-3.073, \gamma_5=-0.920, \gamma_6=0.750$. Thus, $E_{\mathfrak{DD}}(G)\approx 17.315$.

Further, we defined some definitions, that are needed for the later part of this paper, as follows:

For a graph G, the *divisor degree of a vertex* v_i denoted by $dd(v_i)$ or dd_i, is defined in [15] as

$$dd(v_i)=\begin{cases}\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right) & \text{if } v_i \text{ and } v_k \text{ are adjacent}\\ 1 & \text{if } d_i=d_k\text{ ; } v_i \text{ and } v_k \text{ are adjacent}\\ 0 & \text{otherwise}\end{cases}$$

where $[x]$ denotes an integral part of real number x and $\sum\limits_{i\sim k}$ means summation over all pair of adjacent vertices v_i and v_k.

For a graph G, the *divisor degree index* $dd(G)$ is defined in [15] as

$$dd(G)=\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right)\right)=\sum_{i=1}^{n}dd(v_i).$$

For any graph G, we defined in [15] as follows:

(i) $\delta_{dd}(G)=min\,\{dd(v)/v\epsilon V(G)\}$ is called *minimum divisor degree* of G.

$(ii)\Delta_{dd}(G)=max\,\{dd(v)/v\epsilon V(G)\}$ is said to be *maximum divisor degree* of G.

For any graph G, defined in [17] as follows:

(i) The *forgotten topological index* is given by

$$F(G)=\sum_{i=1}^{n} d_i^3=\sum_{v_iv_j\epsilon E(G)}\left(d_i^2+d_j^2\right).$$

(ii) The *modified second Zagreb index* is given by

$$M_2^*(G)=\sum_{v_iv_j\epsilon E(G)}\frac{1}{d_id_j}.$$

2 Some known and related results for bounds of energy

The following is well known and related results for bounds of energy which are needed for the later part of this paper to find the bounds for the divisor degree energy of G.

Lemma 2.1. [24] If $\mathbf{C}$ is a symmetric matrix of order n with non-increasing eigenvalues $\rho_1 \geq \rho_2 \geq ... \geq \rho_n$, then $\mathbf{x}^T\mathbf{C}\mathbf{x} \leq \rho_1\mathbf{x}^T\mathbf{x}$, for any $\mathbf{x}\epsilon R^n - \{0\}$.

Lemma 2.2. [12] Let $A = (a_{ik})$and B = (b_{ik}) be symmetric, non-negative matrices with n vertices. If $B \leq A$, that is$b_{ik} \leq \ a_{ik}$ for all i, k, then $\rho_1(A) \leq \rho_1(B)$, where ρ_1 is the largest eigenvalue.

Lemma 2.3. [11] If G is a simple graph with n vertices and m edges, then $\lambda_1(G) \leq\sqrt{2m-n+1}$ with equality holds if and only if G is a star graph or a complete graph.

Lemma 2.4. [15] Let G be a simple and connected graph with n vertices and m edges. Then $dd(G) \geq 2m$ with equality holds if G is regular.

Theorem 2.5. [16] If G is a connected graph with n vertices and m edges, then

$$2m \leq dd(G) < n(n^2-2n+2).$$

Theorem 2.6. Let G be a graph with n vertices and m edges, then

$$\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right)^2\right) \geq \frac{4m^2}{n} \tag{2}$$

with equality holds if $G \cong \overline{K_n}$.

Proof: By Cauchy-Schwarz inequality,

$$\left(\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right)\right)\right)^2 \leq n\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right)^2\right)$$

By Lemma 2.4, inequality (2) follows.
If $G \cong \overline{K_n}$, then $m = 0$ and so $\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right)^2\right) = 0.$ ∎

3 Bounds for spectral radius of divisor degree matrix of graphs

Theorem 3.1. If G is a simple graph with n vertices, then

$$\gamma_1 < \sqrt{n(n-1)(n-2)}. \tag{3}$$

Proof: Let i^{th} row and i^{th} row sum of $\mathfrak{D}\mathfrak{D}$ be $\mathfrak{D}\mathfrak{D}_i$ and dd_i respectively. Let the eigenvector of $\mathfrak{D}\mathfrak{D}$ with unit length be $\mathbf{X} = (x_1, x_2, ..., x_n)^T$ and its corresponding eigenvalue is $\gamma_1(\mathfrak{D}\mathfrak{D})$. Let the vector $\mathbf{X}(i)$ is obtained from $\mathbf{X}$ for $i = 1, 2, ..., n$, by changing those components x_k by zero such that a_{ik} is zero. Now, for $i = 1, 2, ..., n$, $(\mathfrak{D}\mathfrak{D})\mathbf{X}(i) = \gamma_1\mathbf{X}$, , then

$$\mathfrak{D}\mathfrak{D}_i\,\mathbf{X}(i) = \mathfrak{D}\mathfrak{D}_i\,\mathbf{X} = \gamma_1(\mathfrak{D}\mathfrak{D})\,x_i$$

Using Cauchy-Schwarz inequality,

$$\gamma_1^2(\mathfrak{D}\mathfrak{D})\,x_i^2 = |\mathfrak{D}\mathfrak{D}_i\mathbf{X}(i)|^2 \leq |\mathfrak{D}\mathfrak{D}_i|\,|\mathbf{X}(i)|^2$$
$$\leq dd_i\left(1 - \sum_{k:a_{ik}=0} x_k^2\right)$$

Adding the above inequalities and using Theorem 2.5, we get

$$\gamma_1^2(\mathfrak{D}\mathfrak{D}) \leq \sum_{i=1}^{n} dd_i - \sum_{i=1}^{n} dd_i \sum_{k:a_{ik}=0} x_k^2$$
$$< n(n^2-2n+2) - \sum_{i=1}^{n} dd_i \sum_{k:a_{ik}=0} x_k^2$$

Now

$$\sum_{i=1}^{n} dd_i \sum_{k:a_{ik}=0} x_k^2 \geq \sum_{i=1}^{n} dd_i\,x_i^2 + \sum_{i=1}^{n} dd_i \sum_{k:a_{ik}=0} x_k^2$$

$$\geq \sum_{i=1}^{n} dd_i \; x_i^2 + \sum_{i=1}^{n} (n^2 - dd_i) \; x_i^2 \geq n^2$$

$$\gamma_1^2(\mathfrak{D}\mathfrak{D}) < n(n^2 - 2n + 2) - n^2$$

and inequality (3) follows. ■

Theorem 3.2. If G is a simple graph of order n with maximum divisor degree Δ_{dd}, then

$$\gamma_1 > \frac{F - 2M_2^*}{n \; \Delta_{dd}^2} \tag{4}$$

where F and M_2^* are forgotten topological index and modified second Zagreb index respectively.

Proof: Let the unit vector be $\mathbf{Y} = (y_1, y_2, ..., y_n)^T$, where $\mathbf{Y}$ belongs to R^n. So

$$Y^T \mathfrak{D}\mathfrak{D}(G) Y = \sum_{v_i v_k \epsilon E(G)} \left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right] \right) x_i x_k$$

$$> \sum_{v_i v_k \epsilon E(G)} \left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right] - 2 \right) x_i x_k$$

$$> \sum_{v_i v_k \epsilon E(G)} \left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right] \right) x_i x_k - \sum_{v_i v_k \epsilon E(G)} 2 x_i x_k$$

$$> \sum_{v_i v_k \epsilon E(G)} 2 x_i x_k + \sum_{v_i v_k \epsilon E(G)} \frac{(d_i - d_k)^2}{d_i d_k} x_i x_k - \sum_{v_i v_k \epsilon E(G)} 2 x_i x_k$$

$$> \sum_{v_i v_k \epsilon E(G)} \frac{(d_i - d_k)^2}{d_i d_k} x_i x_k > \frac{F - 2M_2^*}{n \; \Delta^2} > \frac{F - 2M_2^*}{n \; \Delta_{dd}^2}$$

Using Lemma 2.1, inequality (4) follows. ■

Theorem 3.3. Let γ_1 be the spectral radius of divisor degree matrix of a simple graph G with n vertices and m edges. Then

$$\gamma_1 \leq \left[n - 1 + \frac{1}{n-1} \right] \sqrt{2m - n + 1} \tag{5}$$

with equality holds if and only if G is a star graph.

Proof: Let $v_i v_k \epsilon E(G)$.

$$\left(\frac{d_i}{d_k}+\frac{d_k}{d_i}\right)\leq\frac{\Delta}{\delta}+\frac{\delta}{\Delta}\leq n-1+\frac{1}{n-1}$$

which equals if and only if $d_i = n-1, d_k = 1$ or $d_k = n-1, d_i = 1$.
If ρ_1 is the greatest eigenvalue of the matrix $\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right)A(G)$, then by Lemma 2.2, 2.3 and $\gamma_1 \leq \rho_1$, inequality (5) follows. ∎

4 Some lower and upper bounds of divisor degree energy of Graphs

In this section, we obtain some lower and upper bounds for the divisor degree energy $E_{\mathfrak{DD}}$ of graph G.

Theorem 4.1. Let $\mathfrak{DD}$ be the divisor degree matrix of a simple graph $G(n,m)$, with absolute determinant value Δ, then

$$E_{\mathfrak{DD}}(G)\geq\sqrt{\frac{4m^2}{n}+n(n-1)\Delta^{\frac{2}{n}}}. \tag{6}$$

Proof: From Eq. (1) We have $(E_{\mathfrak{DD}}(G))^2 = \left(\sum_{i=1}^n |\gamma_i|\right)^2 = \sum_{i=1}^n |\gamma_i| \sum_{i=1}^n |\gamma_i|$

$$=\sum_{i=1}^{n}\gamma_i^2+\sum_{i\neq k}^{n}|\gamma_i|\,|\gamma_i| \tag{7}$$

We know that for non-negative integer, the geometric mean is not larger than the arithmetic mean,

$$\frac{1}{n(n-1)}\sum_{i\neq j}^{n}|\gamma_i|\,|\gamma_i|\geq\left(\Pi_{i\neq j}|\gamma_i|\,|\gamma_i|\right)^{\frac{1}{n(n-1)}}\geq\left(\left(\Pi_{i\neq j}|\gamma_i|\right)^{2(n-1)}\right)^{\frac{1}{n(n-1)}}=\Delta^{\frac{2}{n}}$$

Eq. (7) becomes,

$$\begin{aligned}(E_{\mathfrak{DD}}(G))^2 &\geq \sum_{i=1}^{n}\gamma_i^2+n(n-1)\Delta^{\frac{2}{n}}\\ &=\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right]+\left[\frac{d_k}{d_i}\right]\right)^2\right)+n(n-1)\Delta^{\frac{2}{n}}\\ &\geq\frac{4m^2}{n}+n(n-1)\Delta^{\frac{2}{n}}\end{aligned}$$

and inequality (6) follows. ∎

Theorem 4.2. For a complete graph K_n, $(n-1)$ is a eigenvalue of divisor degree matrix of K_n and $E_{\mathfrak{DD}}(K_n)\leq E_{\mathfrak{DD}}(K_{1,n-1})$.

Proof: We have

$$|\gamma\, I - \mathfrak{D}\mathfrak{D}(K_n)| = \begin{vmatrix} \gamma & -1 & -1 & ... & -1 \\ -1 & \gamma & -1 & ... & -1 \\ -1 & -1 & \gamma & ... & -1 \\ ... & ... & ... & ... & ... \\ -1 & -1 & -1 & ... & \gamma \end{vmatrix}$$

Now by using elementary operation $C_1 \to C_1{+}C_2{+}...{+}C_n$, we get the factor of $|\gamma\, I - \mathfrak{D}\mathfrak{D}(K_n)|$ is $(\gamma - (n-1))$. Thus $(n-1)$ is a eigenvalue of $\mathfrak{D}\mathfrak{D}(K_n)$.
Since $tr(\mathfrak{D}\mathfrak{D}(K_n))^2 = n(n-1)$, we have

$$(E_{\mathfrak{D}\mathfrak{D}}(K_n))^2 \le n^2(n-1)$$

Also

$$(E_{\mathfrak{D}\mathfrak{D}}(K_{1,n-1}))^2 \le\ 2n(n-1)^3$$

Therefore, $E_{\mathfrak{D}\mathfrak{D}}(K_n) \le E_{\mathfrak{D}\mathfrak{D}}(K_{1,n-1})$. ∎

Theorem 4.3. Let γ_i be any eigenvalue of divisor degree matrix of a simple graph G with n vertices. Then for any *i*, we have $|\gamma_i| \le (n-1)^2\sqrt{\frac{2}{n}}$.

Proof: We have

$$tr(\mathfrak{D}\mathfrak{D}(K_{1,n-1}))^2 = 2(n-1)^3$$

Therefore for any graph G with n vertices, the divisor degree matrix of G has its eigenvalues $\gamma_1, \gamma_2, ..., \gamma_n$, we have

$$\sum_{i=1}^{n} |\gamma_i|^2 \le 2(n-1)^3$$

Using Cauchy-Schwarz inequality, we have

$$\sum_{i\neq k} |\gamma_i|^2 = (n-1)\sum_{i\neq k} |\gamma_i|^2$$

$$\gamma_i^2 \le (n-1)(2(n-1)^3 - \gamma_i^2)$$

Therefore, $|\gamma_i| \le (n-1)^2\sqrt{\frac{2}{n}}$. ∎

Theorem 4.4. For a wheel graph W_n with $n(n \ge 4)$ vertices,
$tr(\mathfrak{D}\mathfrak{D}(W_n))^2 = 2\,(n-1)\left(1 + \left[\frac{n-1}{3}\right]^2\right)$and $E_{\mathfrak{D}\mathfrak{D}}(W_n) < \sqrt{2n(n-1)\left(1 + \left[\frac{n-1}{3}\right]^2\right)}$.

Proof: The divisor degree matrix of W_n is

$$\mathfrak{DD}(W_n) = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 1 & \left[\frac{n-1}{3}\right] \\ 1 & 0 & 1 & ... & 0 & 0 & \left[\frac{n-1}{3}\right] \\ 0 & 1 & 0 & ... & 0 & 0 & \left[\frac{n-1}{3}\right] \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 & 1 & \left[\frac{n-1}{3}\right] \\ 1 & 0 & 0 & ... & 1 & 0 & \left[\frac{n-1}{3}\right] \\ \left[\frac{n-1}{3}\right] & \left[\frac{n-1}{3}\right] & \left[\frac{n-1}{3}\right] & ... & \left[\frac{n-1}{3}\right] & \left[\frac{n-1}{3}\right] & 0 \end{bmatrix}$$

$$tr(\mathfrak{DD}(W_n))^2 = (n-1)\left(2 + \left[\frac{n-1}{3}\right]^2\right) + (n-1)\left[\frac{n-1}{3}\right]^2$$

$$= 2(n-1)\left(1 + \left[\frac{n-1}{3}\right]^2\right)$$

$$\sum_{i=1}^{n} \gamma_i^2 = 2(n-1)\left(1 + \left[\frac{n-1}{3}\right]^2\right)$$

Using Cauchy-Schwarz inequality,

$$\sum_{i=1}^{n} \gamma_i < \sqrt{2n(n-1)\left(1 + \left[\frac{n-1}{3}\right]^2\right)}$$

Hence, $E_{\mathfrak{DD}}(W_n) < \sqrt{2n(n-1)\left(1 + \left[\frac{n-1}{3}\right]^2\right)}$. ∎

Theorem 4.5. For a path graph P_n with $n(n \geq 4)$ vertices, $tr(\mathfrak{DD}(P_n))^2 = 2\,(n+5)$ and $E_{\mathfrak{DD}}(P_n) < \sqrt{2n\,(n+5)}$.

Proof: The divisor degree matrix of P_n is

$$\mathfrak{D}\mathfrak{D}(P_n) = \begin{bmatrix} 0 & 2 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 2 \\ 0 & 0 & 0 & \cdots & 0 & 2 & 0 \end{bmatrix}$$

$$tr(\mathfrak{D}\mathfrak{D}(P_n))^2 = 18 + 2(n-4) = 2(n+5).$$

Using Cauchy-Schwarz inequality,

$$\sum_{i=1}^{n} \gamma_i < \sqrt{2n(n+5)}$$

Therefore,$E_{\mathfrak{D}\mathfrak{D}}(P_n) < \sqrt{2n(n+5)}$ $(n \geq 4)$. ■

Theorem 4.6. If P_n is path graph of order $n(n \geq 4)$, then $E_{\mathfrak{D}\mathfrak{D}}(L(P_n)) < \sqrt{2(n-1)(n+4)}$ where $L(P_n)$ is a line graph of P_n.

Proof: If P_n is a path graph of order n, then $P_n - e$ is the corresponding line graph of P_n of order $n-1$. That is, $L(P_n) \cong P_{n-1}$.
Hence by using Theorem 4.5, we get $E_{\mathfrak{D}\mathfrak{D}}(L(P_n)) < \sqrt{2(n-1)(n+4)}$. ■

Theorem 4.7. If S_n is star graph of order n, then $E_{\mathfrak{D}\mathfrak{D}}(L(S_n)) = 2(n-2)$where $L(S_n)$ is a line graph of S_n.

Proof: If S_n is a star graph of order n, then the vertex u is adjacent to $n-1$ vertices, which means u has an edge incident with every other $n-1$ vertices. Thus the corresponding line graph of S_n is a complete graph of order $n-1$. That is, $L(S_n) \cong K_{n-1}$.
Hence $E_{\mathfrak{D}\mathfrak{D}}(L(S_n)) = 2(n-2)$, where $E_{\mathfrak{D}\mathfrak{D}}(K_{n-1}) = 2(n-1)$. ■

Theorem 4.8. If $G(n, m)$ is a simple graph. Then

$$E_{\mathfrak{D}\mathfrak{D}}(G) \leq \frac{2m}{n} + \sqrt{(n-1)\left\{\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)^2\right) - \frac{4m^2}{n^2}\right\}}. \tag{8}$$

Proof: We have

$$E_{\mathfrak{D}\mathfrak{D}}(G) = \sum_{i=1}^{n} |\gamma_i| = \gamma_1 + \sum_{i=2}^{n} |\gamma_i|$$

Using Cauchy-Schwarz inequality,

$$\sum_{i=2}^{n} |\gamma_i| \leq \sqrt{(n-1)\sum_{i=2}^{n} \gamma_i^2}$$

$$E_{\mathfrak{D}\mathfrak{D}}(G) \leq \gamma_1 + \sqrt{(n-1)\left\{\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)^2\right) - \gamma_1^2\right\}}$$

Set $\gamma_1 = x$. Define the function

$$f(x) = x + \sqrt{(n-1)\left\{\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)^2\right) - x^2\right\}}$$

From

$$\sum_{i=1}^{n} \gamma_i^2 = \sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)^2\right)$$

we get,

$$x^2 = \gamma_1^2 \leq \sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)^2\right)$$

$$x \leq \sqrt{\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)^2\right)}$$

Now, $f'(x) = 0$ implies,

$$x = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)^2\right)}$$

Therefore, the interval of a decreasing function $f(x)$ is

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)^2\right)} \leq x \leq \sqrt{\sum_{i=1}^{n}\left(\sum_{i\sim k}\left(\left[\frac{d_i}{d_k}\right] + \left[\frac{d_k}{d_i}\right]\right)^2\right)}$$

Using Theorem 2.6, we get

$$\gamma_1 \geq \sqrt{\frac{4m^2}{n^2}} = \frac{2m}{n}$$

$$f(\gamma_1) \leq f\left(\frac{2m}{n}\right)$$

and inequality (8) follows. ■

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