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Composite Labelling of Graphs

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Abstract

In this paper, we introduce the concept of composite labelling. All the graphs considered are undirected connected simple graphs with order n and size m. Let $u, v, w \in$ V(G). A composite labelling is a bijective function f: (V(G) $\cup$ E(G)) $\rightarrow$ $\{1, 2, 3, ..., m + n\}$ such that gcd(f(uv), f(vw)) $\neq 1$. A graph that admits composite labelling is known as a composite graph. We investigate composite labelling for some families of graphs and obtain certain labelling bounds for composite graphs.

Keywords: Graph Labelling, Composite Labelling, Prime Labelling, Total Labelling, Composite Graph

2010 Mathematics Subject Classification : 05C40, 05C69

1 Introduction

We consider undirected connected simple graph G=(V, E) with vertex set V(G) and edge set E(G). Let |V(G)| be denoted by n and |E(G)| be denoted by m, known as the order and size of G respectively. Throughout this paper, T_n denotes a tree on n vertices and C_n denotes a cycle on n vertices. For graph labelling concetps, please see Gallian.[1]

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Definition 1.1. Graph labelling is an assignment of labels to vertices or edges or to both subject to certain conditions.

The concept of prime labeling originated with Entringer and it was later introduced by Tout, Dabbousy and Howalla.[1] They defined *prime labeling* for a graph G with vertex set V, as a bijective function f: V(G) $\to \{1, 2, ..., p\}$ where p is the number of vertices such that each vertex receives a distinct integer from $\{1, 2, ..., p\}$ and for each edge xy the labels assigned to x and y are relatively prime.

Vaidya and Prajapathy[4] introduced the concept of *k*-prime labelling. Ramasubramanian and Kala[6] introduced the concept of *total prime graph.* They defined total prime labelling as a bijective function f: V(G)$\cup$ E(G) $\to \{1, 2, 3, ..., m + n\}$ such that for each edge uv, the labels assigned to vertices u and v are realtively prime and for each vertex of degree at least 2, the greatest common divisor of the labels on the incident edges is 1.

In this paper, we introduce a labelling known as composite labelling. We identify a few families of graphs which admit composite labelling.

Definition 1.2. Let G=(V, E) be an undirected connected simple graph of order n and size m. Let *u, v, w* $\in$ V(G). A composite labelling is a bijective function f: V(G)$\cup$E(G)$\to$ $\{1, 2, 3, ..., m + n\}$ such that gcd(f(uv), f(vw)) $\neq 1$.

It can easily be seen that the labelling introduced here is a variation of *total labelling.* However, in the composite labelling, the stress is given to the edge labelling even though labels of vertices are crucial in labelling the edges.

Definition 1.3. A graph that admits composite labelling is known as a composite graph.

It would be interesting to obtain a bound for the value of $m + n$ so that the graph admits composite labelling.

Theorem 1.4. For any simple connected undirected graph with n vertices m edges admitting composite labelling we have, $2n - 1 \leq$ m+n $\leq \frac{n(n+1)}{2}$.

Proof: In order to obtain the upper bound, let us consider a complete graph whose number of edges is given by ${}^nC_2$. Therefore, $m + n \leq \frac{n(n+1)}{2}$.
For obtaining the lower bound, we consider a tree with n vertices and $n - 1$ edges. We get $2n - 1 \leq m + n$. Therefore, we have, $2n - 1 \leq m + n \leq \frac{n(n+1)}{2}$.

∎

Now we identify certain families of graphs admitting composite labelling.

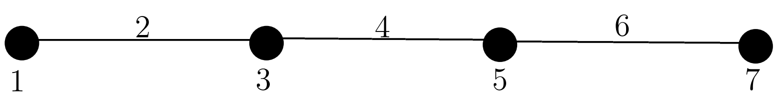
Theorem 1.5. All trees admit composite labelling.

Proof: Trees are acyclic connected graphs with n vertices and $n - 1$ edges. We have to consider the following cases.
Case 1. n is even.

When n is even, then $n - 1$ is odd. Therefore $m + n = 2n - 1$, which is odd. We can label the vertices of a tree with odd labels and edges with even labels so that composite labelling is obtained by trees.
Case 2. n is odd.

When n is odd, $n - 1$ is even. The proof is similar to that of **Case 1**. ∎

The next digram is the composite labelling of P_4.



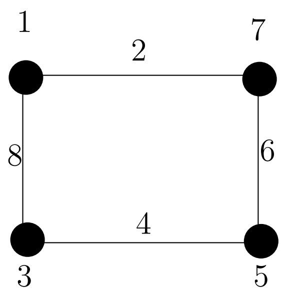
P_4 with composite labelling

Theorem 1.6. All cycles admit composite labelling.

Proof: Let C_n denote a cycle on n vertices. Cycles have equal number of vertices and edges. Therefore $m + n = 2n$, which is even. In the case of cycles $m + n = 2n$, which is always even.

Labelling the vertices with odd labels and edges with even labels produce the composite labelling in the case of cycles.

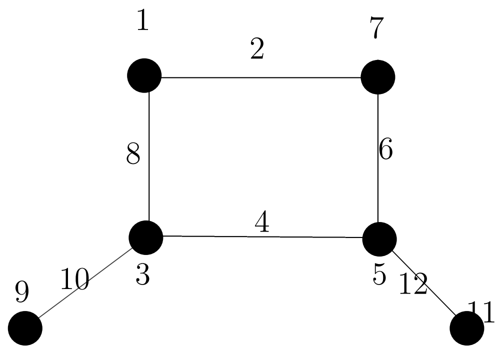
■



C_4 with composite labelling

Theorem 1.7. All unicyclic graphs are composite graphs.

Proof: A unicyclic graph has equal number of vertices and edges. Hence, the the proof is similar to that of theorem 1.6. ■

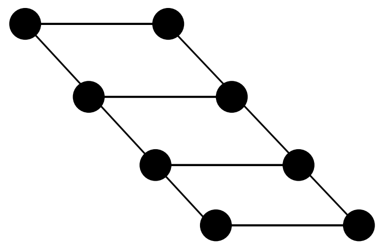


A unicyclic graph with composite labelling

2 Main Result

In this section, we discuss the composite labelling of ladder graphs.

Definition 2.1. Let P_n denote the path on n vertices, then the Cartesian product P_n X P_m, where $m \leq n$, is called a grid graph. If $m = 2$, then the graph is called a ladder, denoted by L_n.



The ladder graph $L_4 = P_4$ X P_2

We, now, prove that,

Theorem 2.2. The ladder graph L_n, i.e., P_n X P_2 is composite.

Proof: Let L_n be the n^{th} ladder P_n X P_2 $\forall n \in \mathbb{N}$. Then L_n has $2n$ vertices and $3n$-2 edges. Let S=$\{1, 2, \ldots 5n\text{-}2\}$.

The ladder graphs L_1 is P_2 and L_2 is C_4. Their composite labellings are discussed in the theorems 1.5 and 1.6, respectively. Both of them accept composite labelling.

Hence, we consider $n \geq 3$.

We use all the integers in S to label the vertices of L_n.

If n is odd, then S has $\frac{(5n-1)}{2}$ odd integers and $\frac{(5n-3)}{2}$ even integers. There is an extra odd integer in S.

If n is even, then S has $\frac{(5n-2)}{2}$ each odd and even integers.

As $2n < \frac{(5n-3)}{2} < \frac{(5n-2)}{2} < \frac{(5n-1)}{2} < 3n$-2, vertices of L_n can be labeled with odd integers always. But, not all edges can be labeled with even integers alone. We require some odd integers as well to complete the composite labeling.

For this, we use the odd multiples of 3.

This is done as follows. Let S_e and S_o be the even an odd integers in S respectively.

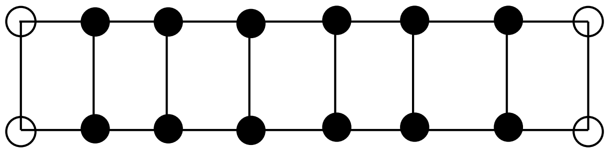
Let $S_{3e} = \{x \in S_e \mid x \text{ is a multiple of } 3\}$. Let $S_{n3e} = S_e \setminus S_{3e}$.

Let $S_{3o} = \{x \in S_o \mid x \text{ is a multiple of } 3\}$. Let $S_{n3o} = S_o \setminus S_{3o}$.

Then $|S_{3e}| = \lfloor \frac{(5n-2)}{6} \rfloor$ and $|S_{n3e}| = \lfloor \frac{(5n-2)}{2} \rfloor$-$\lfloor \frac{(5n-2)}{6} \rfloor$.

Let the above quantities be p and q respectively.

Every ladder has four vertices each of degree 2. Two of them are on the top of the ladder and the other two are at the bottom of the ladder. See the next diagram for more clarity.



A ladder with the initial and terminal steps with specified vertices

From the one side of the ladder with an edge having 2 degree each on either sides, we assign the even integers of S_{n3e} sequentially in the ascending order. Once, q edges are assigned the labels, we continue to assign the p integers of S_{3e} with no edges left out in between.

Hence, a total of $p + q = \lfloor \frac{(5n-2)}{2} \rfloor$ edges are labeled.

When n is odd, there are $\frac{(n-1)}{2}$ edges left unlabelled and when n is even, there are $\frac{(n-2)}{2}$ edges left unlabelled.

As the last labeled edges are multiples of 6, irrespective of the parity of n, we can label the remaining unlabelled edges with the odd multiples of 3. The number of odd multiples of 3 in S is $\lfloor \frac{(5n+1)}{6} \rfloor$. For every $n \geq 3$, $\frac{(n-2)}{2} < \frac{(n-1)}{2} < \lfloor \frac{(5n+1)}{6} \rfloor$).

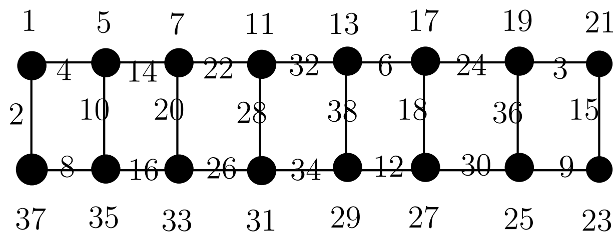
Hence, we can choose any of $\frac{(n-1)}{2}$ or $\frac{(n-2)}{2}$ odd multiples from S_o and assign them as labels to the remaining edges. In addition to these, all the unassigned odd integers of S_o can be assigned to the $2n$ vertices of the ladder graph.

∎

We now see the composite labelling of the ladder graph L_8.

L_8 has 16 vertices and 22 edges. Hence the labels are S={1, 2, ... 38}. The othe associated sets are: $S_{n3e} = \{2, 4, 8, 10, 14, 16, 20, 22, 26, 28, 32, 34, 38\}$. $S_{3e} = \{6, 12, 18, 24, 30, 36\}$. The odd integers taken from S_{3o} are 3, 9 and 15. All the remaining odd integers of S_o are used to label the vertices.

We start the labelling of edges with an edge that is between a vertices of exactly two degrees each. Labels are taken from S_{n3e}, S_{3e}, S_{3o} and S_o in that order. The next diagram is the labelled ladder graph L_8.



The ladder graph L_8 with composite labelling

3 Conclusion

This is only an introductory study. It requires tedious computation and logical skills for checking whether an arbitrary graph is composite or not. Planar graphs have great tendency to be composite. When vertices of larger degrees exist in a graph, the difficulty of finding the composite labelling increases. Hence, they provide a larger scope for strong research in this area.

Acknowledgement

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