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Construction of Graphs with given Ascending Domination Decomposition

Selvam Avadayappan, M. Bhuvaneshwari and P. ChandraDevi
Research Department of Mathematics
VHNSN College, Virudhunagar - 626001, Tamilnadu, India.
selvam_avadayappan@yahoo.co.in
bhuvaneshwari@vhnsnc.edu.in
chandradevi76@gmail.com

Abstract

An Ascending Domination Decomposition (ADD) of a graph G is a collection $\psi = \{G_1, G_2, ..., G_k\}$ of subgraphs of G such that each G_i is connected, every edge of G is in exactly one G_i and domination number of G_i is i, for $1 \leq i \leq k$. In this paper, we prove that every graph G is an induced subgraph of a graph H that admits ADD with $\psi = \{G_1, G_2, ..., G_p\}$ for any positive integer p.

Keywords: Domination, Decomposition, Ascending Domination Decomposition, Splitting graph.
2010 Mathematics Subject Classification: 05C69, 05C70.

1 Introduction

By a graph $G(V, E)$, we mean a non-trivial, finite, simple, undirected and connected graph. The order of a graph is denoted by n. For basic terminology and notations, we refer [3]. Let P_n and C_n denote the path and cycle on n vertices respectively.

A *subdivision* of a graph G is a graph obtained by inserting a new vertex in each edge of G and is denoted by $Sd(G)$.

For example, a graph G and its subdivision graph $Sd(G)$ are given in $Figure$ 1.

* Corresponding Author: *Selvam Avadayappan*
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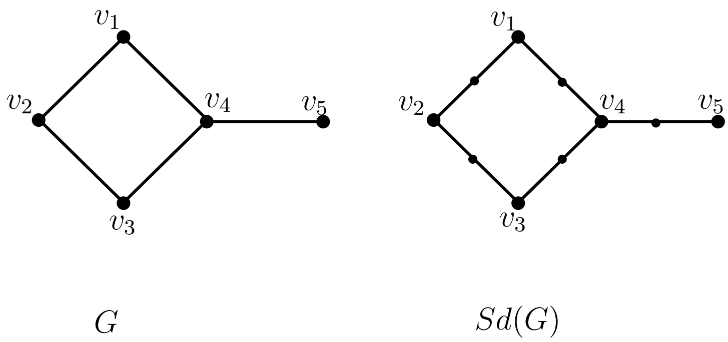


Figure 1

A *wounded spider* is a tree obtained from a star $K_{1,t}$ where $t \geq 2$, by subdividing i edges (where $1 \leq i \leq t-1$) of the star. It is denoted by $WS_{i,t}$.

Let $H_{n,n}$ be the *bipartite* graph with vertex set $\{v_1, v_2, ..., v_n \ ; u_1, u_2, ...\ ., u_n\}$ and the edge set $\{v_i u_j / 1 \leq i \leq n, n-i+1 \leq j \leq n\}$.

For example, the graph $H_{5,5}$ is shown in $Figure$ 2.

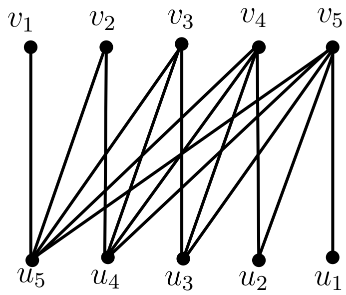


Figure 2

The *corona* $G_1 \circ G_2$ of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 which has p_1 vertices and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2. The graph $G \circ K_1$ is denoted by G^+.

For example, the graphs G_1, G_2 and $G_1 \circ G_2$ are shown in $Figure$ 3.

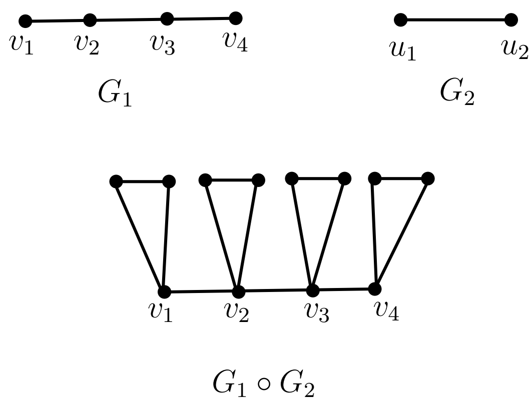


Figure 3

A graph which contains a hamilton path is called a *traceable graph.*

The concept of splitting graph $S(G)$ was introduced by Sampath Kumar and Walikar [8]. The graph $S(G)$ obtained from G, by adding a new vertex w for every vertex $v \in V$ and joining w to all vertices adjacent to v in G, is called the *splitting graph* of G.

For example, a graph G and its splitting graph $S(G)$ are shown in $Figure$ 4.

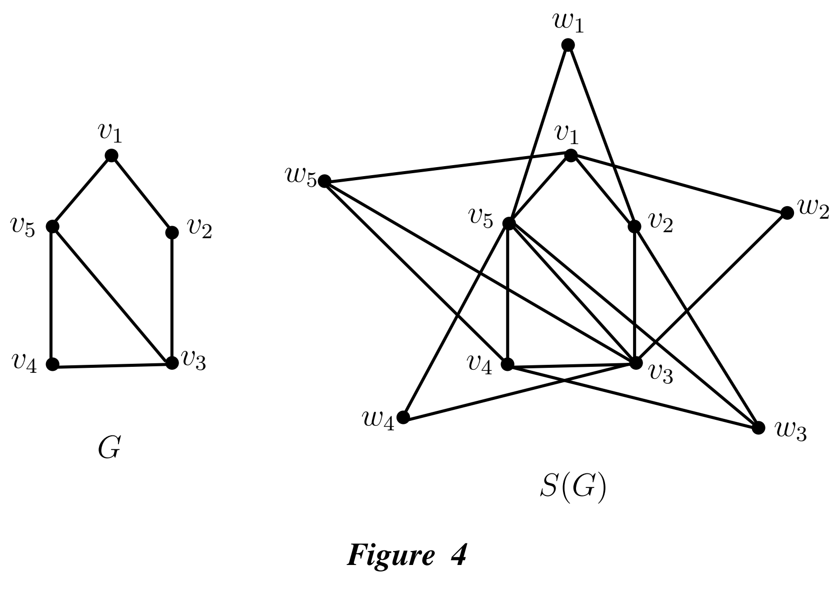


Figure 4

For further study on various types of splitting graphs one can refer [1, 2, 8].

In a graph, a *dominating set* is a subset S of the vertex set such that every vertex is either in S or adjacent to a vertex in S. The *domination number* $\gamma(G)$, is the minimum cardinality among all dominating sets of G. Any dominating set with $\gamma(G)$ vertices is called a $\gamma-set$ of G. The notation $\gamma(G)$ was first used by E.J.Cockayne and S.T.Hedetniemi [5]. A *decomposition* of a graph G is a collection ψ of edge disjoint subgraphs $G_1, G_2, ..., G_n$ of G such that every edge of G is in exactly one G_i. For further results on decomposition one can refer [6].

An *Ascending Domination Decomposition* (ADD) of a graph G is a collection $\psi = \{G_1, G_2, ..., G_k\}$ of subgraphs of G such that
i)each G_i is connected
ii)every edge of G is in exactly one G_i
iii)$\gamma(G_i) = i$, for each i, $1 \leq i \leq k$.
If ψ is an ADD of a graph G, then we say that G admits an ADD.

For example, a graph G with an ADD $\psi = \{G_1, G_2, G_3\}$ are shown in *Figure* 5.

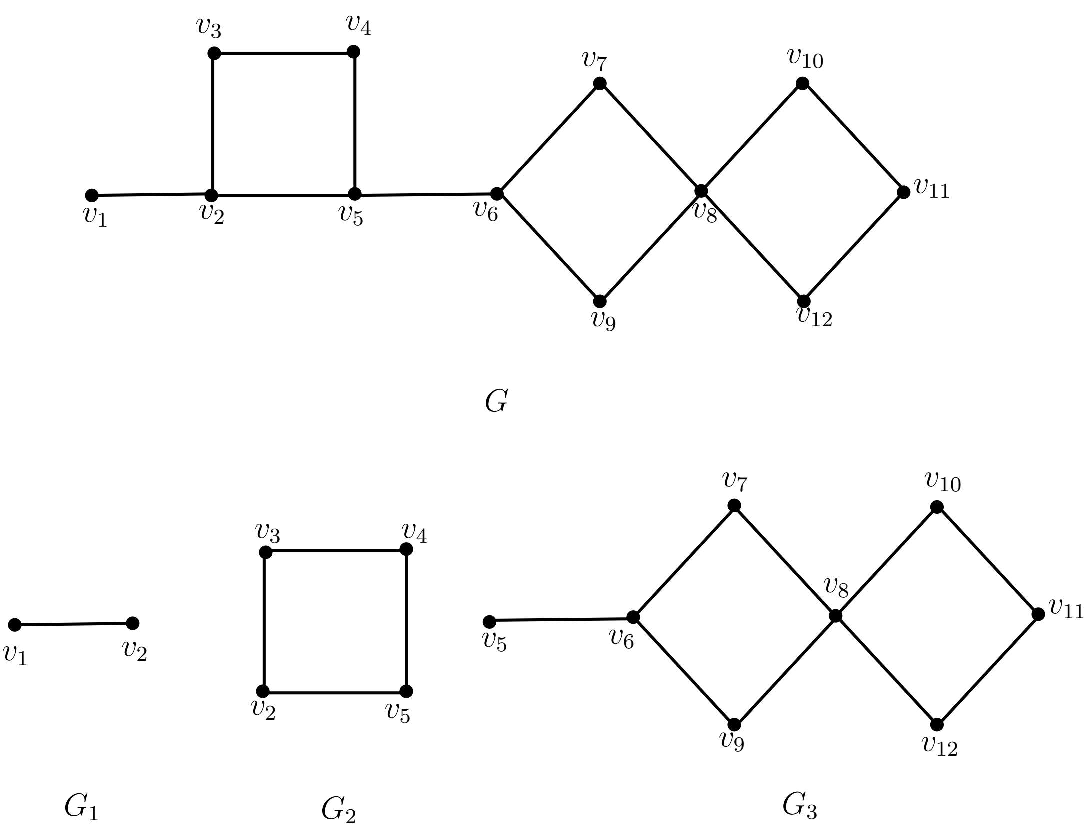


Figure 5

In [7], it has been proved that the graphs K_n, W_n, $K_{1,n}$, $K_{m,n}$, P_n, C_n and the corona graphs P_p^+, C_p^+ and $Sd(K_{1,p})$ admit ADD.

For more results on ADD, one can refer [4].

In this paper, we discuss about ADD in splitting graphs. Also we prove that for any given integer p, there exists a graph with p components of ADD form. More over, we prove that if G is any traceable graph of order $\frac{n(n+1)}{2}$, then G^+ admits an ADD. In addition, we construct a graph H with any given graph G, as an induced subgraph that admits ADD with given number of components.

2 Main Results

We first prove the existence of a graph for any integer $p \geq 1$, which is decomposed into p subgraphs of ADD form.

Theorem 2.1. For any given integer p ≥ 1, there exists a graph which is decomposed into p subgraphs of ADD form.

Proof: Construct a graph G as follows: Let $V(G) = \{v_i, w_{ij}, u_{ij}; 2 \leq i \leq p, 1 \leq j \leq i-1, v_1, w_{11}, u_{11}\}$ and $E(G) = \{v_1w_{11}, w_{11}u_{11}, v_iv_{i+1}; 1 \leq i \leq p-1, v_iw_{ij}, w_{ij}u_{ij}; 2 \leq i \leq p, 1 \leq j \leq i-1\}$ be respectively the vertex set and the edge set of G. Let G_1 be the edge induced subgraph induced by the edges incident with w_{11}. Then $G_1 \cong K_2$ and $\gamma(G_1) = 1$. Let G_2 be the path $v_1v_2w_{21}u_{21}$. Clearly $\gamma(G_2) = 2$. In general for $i \geq 2$, let G_i be the edge induced subgraph induced by the edge set $\{v_iv_{i-1}, v_iw_{ij}, w_{ij}u_{ij}; 2 \leq i \leq p, 1 \leq j \leq p\}$. Clearly each G_i is isomorphic to a wounded spider $WS_{i-1,i}$ and hence connected. Also for any $i \geq 2$, the γ - set of $G_i = \{v_i, w_{ij}/1 \leq j \leq i-1\}$ and thus $\gamma(G_i) = i$. Therefore $\psi = \{G_1, G_2, G_3, ..., G_p\}$ is the ADD form of G with p components. ∎

The case when $p = 4$ is illustrated in *Figure* 6.

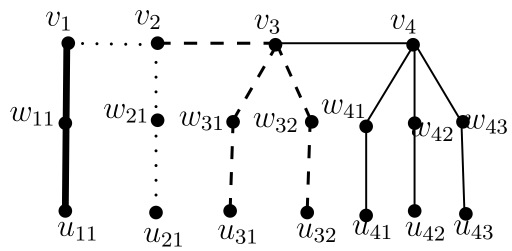


Figure 6 : G

Here the edges of G_1, G_2, G_3 and G_4 are drawn with bold lines, dotted lines, dashed lines and slim lines respectively

The constructed graph is not the only family with the said conditions. In fact, for any integer $p \geq 1$, the subdivision graph $Sd(H_{p,p})$ proves the existence of another family with given properties.

For example, the graph $Sd(H_{5,5})$ is shown in *Figure* 7.

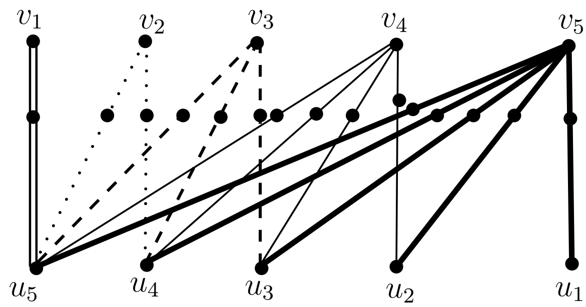


Figure 7 : $Sd(H_{5,5})$

Here the edges of G_1, G_2, G_3, G_4 and G_5 are drawn with double lines, dotted lines, dashed lines, slim lines and bold lines respectively

Theorem 2.2. For any traceable graph G of order $\dfrac{n(n+1)}{2}$, $G \circ K_1$ admits ADD.

Proof: Let G be a traceable graph of order $p = \dfrac{n(n+1)}{2}$. Then G has a hamilton path $v_1v_2....v_p$. Let $H = G \circ K_1$. Let G_1 be an edge induced subgraph induced by the edges incident with v_1. Therefore $\gamma(G_1) = 1$. Let G_2 be an edge induced subgraph induced by the edges incident with $\{v_2, v_3\}$ which are not in G_1. Clearly G_2 is connected with $\gamma(G_2) = 2$. In general for $i \geq 2$, let G_i be an edge induced subgraph induced by the edges incident with $\left\{ v_{\frac{i(i-1)}{2}+1}, ..., v_{\frac{i(i+1)}{2}} \right\}$ excluding the edges in $\bigcup_{j=1}^{i-1} E(G_j)$. Since G has a hamilton path, each G_i is connected. Now each v_i is incident with a pendant vertex in H, $\left\{ v_{\frac{i(i-1)}{2}+1}, ..., v_{\frac{i(i+1)}{2}} \right\}$ is a $\gamma-$ set of G_i. That is, $\gamma(G_i) = i$. Hence $\psi = \{G_1, G_2, ..., G_p\}$ admits ADD. ∎

As an illustration, a traceable graph G of order 6 admitting an ADD is given in *Figure* 8.

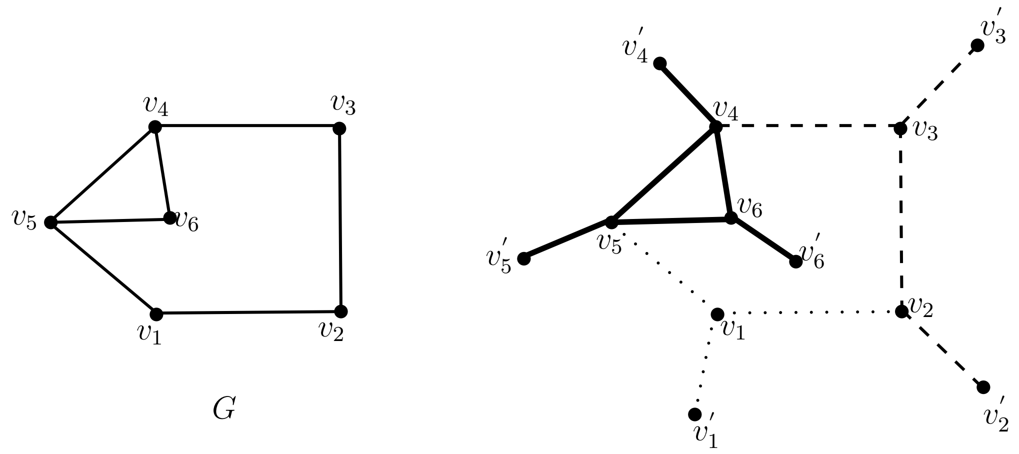


Figure 8

Here the edges of G_1, G_2 and G_3 are drawn with dotted lines, dashed lines and bold lines respectively

Theorem 2.3. Any graph G of order n is an induced subgraph of a graph H with ADD $\psi = \{G_1, G_2, ..., G_n\}$.

Proof: Let G be any graph of order n. Let $V(G) = \{v_0, v_1, ..., v_{n-1}\}$. Take $V(H) = V(G) \cup \{u_i, u_{ik}/1 \leq i \leq n-1, 1 \leq k \leq 2i\}$ and $E(H) = E(G) \cup \{v_i u_i, v_i u_{ij}/1 \leq i \leq n-1; 1 \leq j \leq i; u_{ij} u_{i(j+i)}/1 \leq j \leq i; 1 \leq i \leq n-1\}$. Let G_1 be an edge induced subgraph induced by the edges incident with v_0. Then $\gamma(G_1) = 1$. Let G_2 be an edge induced subgraph induced by the edges incident with $\{v_1\}$ which are not in G_1. Hence G_2 is connected with $\gamma(G_2) = 2$. In general, Let G_i be an edge induced subgraph induced by the edges incident with v_{i-1} excluding the edges in $\cup_{j=1}^{i-1} E(G_j)$. Each G_i is isomorphic to a WS_i for some $r \geq i+1$. Hence every G_i is connected with a γ - set

$\{v_{i-1}, u_{i-1,j}/1 \leq j \leq i-1\}$ and thus $\gamma(G_i) = i$. Therefore $\psi = \{G_1, G_2, ..., G_n\}$ is an ADD for the graph H. It is clear that G is an induced subgraph of H. ∎

For example, the graph H which contains $G = C_5$ as an induced subgraph with ADD $\psi = \{G_1, G_2, G_3, G_4, G_5\}$ is shown in *Figure* 9.

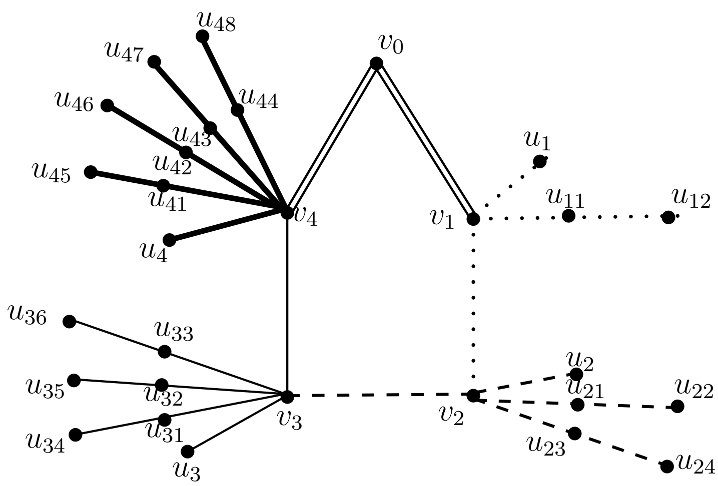


Figure 9

Here the edges of G_1, G_2, G_3, G_4 and G_5 are drawn with double lines, dotted lines, dashed lines, slim lines and bold lines respectively

Even more, the following theorem proves that every graph of order n is an induced subgraph of a graph of ADD with p components, for any given $p < n$.

Theorem 2.4. Any graph G of order n is an induced subgraph of a graph H with $\psi = \{G_1, G_2, ..., G_p\}$, where $2 \leq p < n$.

Proof: Let G be any graph of order n. Let $V(G) = \{v_0, v_1, ..., v_{n-1}\}$. Construct the graph H as follows: Let $V(H) = V(G) \cup \{u_i, u_{ij}/1 \leq i \leq p-1, 1 \leq j \leq 2i\}$ and $E(H) = E(G) \cup \{v_i u_i, v_i u_{ij}/1 \leq i \leq p-1, 1 \leq j \leq i, u_{ij} u_{i,j+i}/1 \leq i \leq p-1, 1 \leq j \leq i\}$. Let G_1 be an edge induced subgraph induced by the edges incident with v_0. Then $\gamma(G_1) = 1$. Let G_2 be an edge induced subgraph induced by the edges incident with $\{v_1\}$ which are not in G_1. Clearly $\gamma(G_2) = 2$. In general for $i \geq 2$, let G_i be an edge induced subgraph induced by the edges incident with v_{i-1} excluding the edges in $\cup_{j=1}^{i-1} E(G_j)$. G_i is connected with a $\gamma-$ set $\{v_{i-1}, u_{i-1,j-1}/2 \leq j \leq i\}$ and thus $\gamma(G_i) = i$. Hence $\psi = \{G_1, G_2, ..., G_p\}$ is an ADD for G. ∎

For example, a graph G on 5 vertices and the corresponding ADD graph H with $p = 3$ having G as induced subgraph are shown in $Figure$ 10.

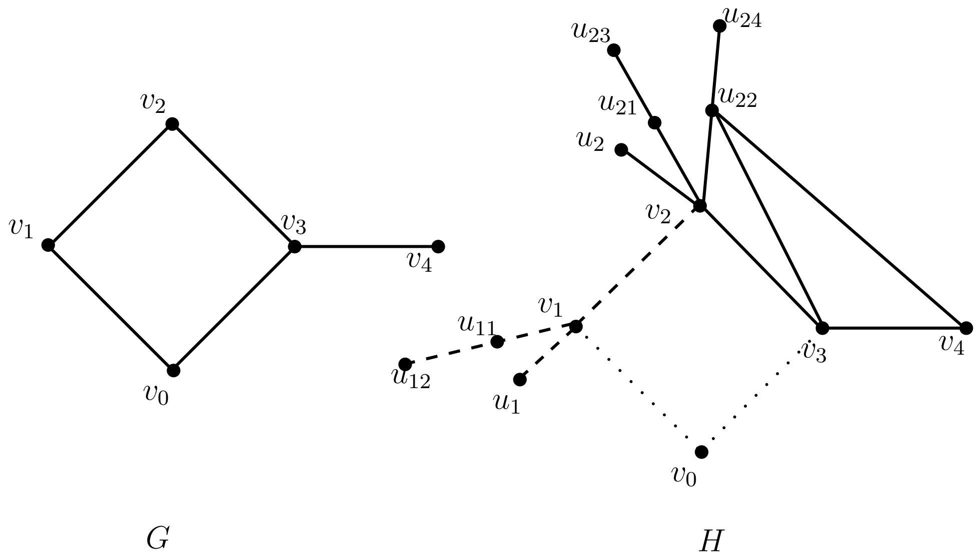
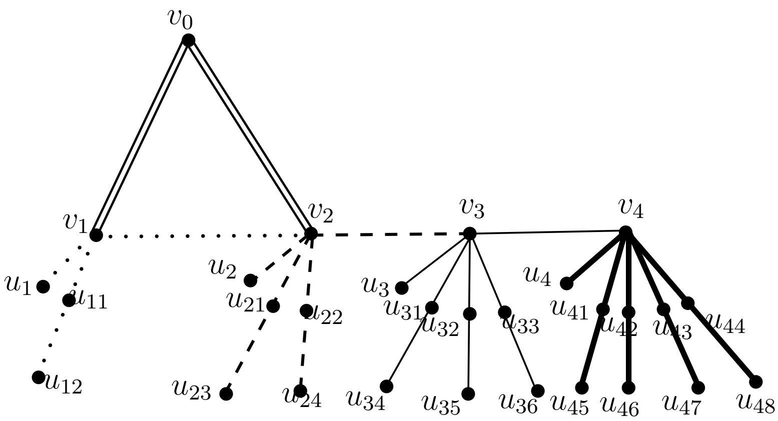


Figure 10
Here the edges of G_1, G_2 *and* G_3 *are drawn with dotted lines, dashed lines and slim lines respectively*

Theorem 2.5. Any graph G of order n is an induced subgraph of a graph H with ADD $\psi = \{G_1, G_2, ..., G_p\}$, where $p > n$.

Proof: Let G be any graph of order n. Let $V(G) = \{v_0, v_1, ..., v_{n-1}\}$. Construct the graph H as follows: Let $V(H) = V(G) \cup \{v_m/n \leq m \leq p-1\} \cup \{\ u_i, u_{ik}/1 \leq i \leq p-1, 1 \leq k \leq 2i\}$ and $E(H) = E(G) \cup \{v_i u_{ij},\ u_{ij}u_{i,j+i}\ /1 \leq i \leq p-1,\ 1 \leq j \leq i\}$. Let G_1 be an edge induced subgraph induced by the edges incident with v_0. Then $\gamma(G_1) = 1$. Let G_2 be an edge induced subgraph induced by the edges incident with $\{v_1\}$ which are not in G_1. Clearly $\gamma(G_2) = 2$. In general, Let G_p be an edge induced subgraph induced by the edges incident with v_{p-1} excluding the edges in $\cup_{j=1}^{i-1} E(G_j)$. Each G_i is isomorphic to a $WS_{i,r}$ for some r $\geq$ i+1. Hence every G_i is connected with a $\gamma-$ set $\{v_{i-1}, u_{i-1,j}/1 \leq j \leq i-1\}$ and $\gamma(G_i) = i$. Hence $\psi = \{G_1, G_2, ..., G_p\}$ is an ADD for G. ∎

As an illustration, for the graph C_3, the corresponding ADD graph H is given in $Figure$ 11.



H

Figure 11

Here the edges of G_1, G_2 G_3, G_4 *and* G_5 *are drawn with double lines, dotted lines, dashed lines, slim lines and bold lines respectively*

Combining theorems 2.3, 2.4 and 2.5, we state the following theorem.

Theorem 2.6. For any given $p \geq 1$, and for any graph G there exists an ADD graph H which contains G as an induced subgraph with $\psi = \{G_1, G_2, ..., G_p\}$.

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