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Directed Pathos Block Vertex Digraph of an Arborescence

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Abstract

A *directed pathos block vertex digraph* of an arborescence A_r, written $Q = DPBV(A_r)$, is the digraph whose vertex set $V(Q) = V(A_r) \cup B(A_r) \cup P(A_r)$, where $B(A_r)$ is the set of blocks (arcs) and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: for every $v \in V(A_r)$, all arcs $(B_1, v); (v, B_2)$; for which v is a head of the block B_1 and tail of the block B_2 in A_r; (P, B) such that $B \in B(A_r)$ and $P \in P(A_r)$ and the block B lies on the directed path P; (P_i, P_j) such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j. The problem of reconstructing an arborescence from its directed pathos block vertex digraph is discussed. The characterization of digraphs whose $DPBV(A_r)$ are planar; outerplanar; maximal outerplanar; and minimally nonouterplanar is presented.

Keywords: Line digraph, directed path number, crossing number, inner vertex number.

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1 Introduction

For graph and digraph theoretical terminologies and notations in this paper we follow the books [1,4]. There are many (di)graph valued functions (or (di)graph operators) with which one can construct a new (di)graph from a given (di)graph, such as the line (di)graph, the middle (di)graph, the block (di)graph, and their generalizations.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common. Harary and Norman [3] extended the concept of line graph of a graph and introduced the concept of line digraph of a directed graph. The *line digraph* $L(D)$ of a digraph $D = (V, A)$ has the arcs of D as vertices. There is an arc from D-arc pq towards D-arc uv if and only if $q = u$.

Hamada and Yoshimura [5] defined a graph $M(G)$ as an intersection graph $\Omega(G)$ on the vertex set $V(G)$ of any graph G. Let $E(G)$ be the edge set of G and $F = V'(G) \cup E(G)$, where $V'(G)$ indicates the family of one-vertex subsets of the set $V(G)$. Let $M(G) = \Omega(G)$. $M(G)$ is called the *middle graph* of G.

Zamfirescu [9] extended the concept of middle graph of a graph to the directed case there by introducing a digraph operator called the *middle digraph*. The *middle digraph* $M(D)$ of a directed graph $D = (V, A)$ is obtained from $L(D)$ by adding all vertices of V, and, for every $v \in V(D)$, all arcs $(a_1, v), (v, a_2)$; for which v is a head of the arc a_1 and tail of the arc a_2 in D.

Kulli [7] introduced the concept of block-vertex graph of a graph. The *block-vertex graph* $BV(G)$ of a graph G is the graph whose vertices can be put in one-to-one correspondence with the set of vertices and blocks of G in such a way that two vertices of $BV(G)$ are adjacent if and only if one corresponds to a block B of G and the other to a vertex v of G and v is in B.

The concept of pathos of a graph G was introduced by Harary [2] as a collection of minimum number of edge disjoint open paths whose union is G. The *path number* of a graph G is the number of paths in any pathos. The *path number* of a tree T equals k, where $2k$ is the number of odd degree vertices of T.

Nagesh [8] introduced the concept of pathos block vertex graph of a tree. A *pathos block vertex graph* of a tree T, written $PBV(T)$, is a graph whose vertices are the vertices, blocks (edges), and paths of a pathos of T, with two vertices of $PBV(T)$ adjacent whenever one corresponds to a block B_i of T and the other to a vertex v_j of T such that B_i is incident with v_j or the block lies on the corresponding path of the pathos; two distinct pathos vertices P_m and P_n of $PBV(T)$ are adjacent whenever the corresponding paths of the pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex in T.

Note that there is freedom in marking the paths of a pathos of a tree T in different ways, provided that the path number k of T is fixed. Since the order of marking the paths of a pathos of a tree is not unique, the corresponding pathos block vertex graph is also not unique. See Figure 1 for an example of a tree and its pathos block vertex graph.

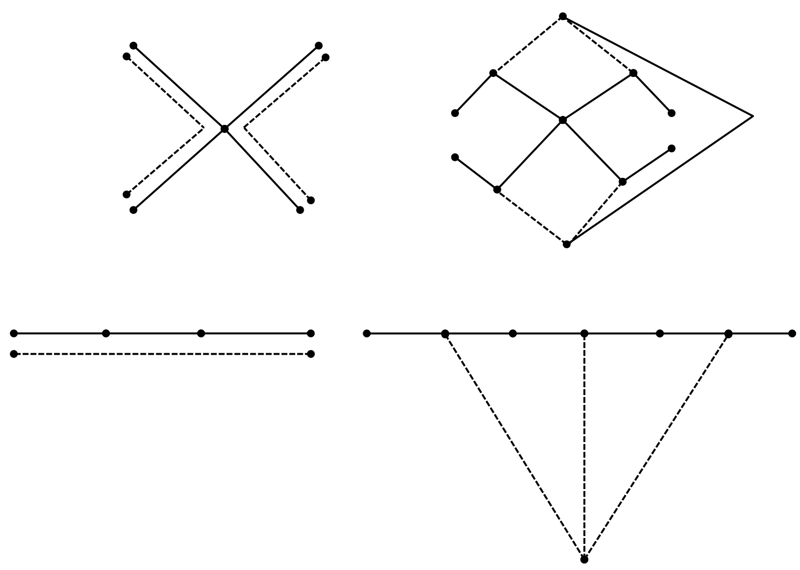

Figure 1

Motivated by the studies above, we extend the concept of pathos block vertex graph of a tree to the directed case and introduce a digraph operator called a *directed pathos block vertex digraph* of an arborescence and develop some of its properties.

2 Preliminaries

We need some concepts and notations on graphs and directed graphs. A *graph* $G = (V, E)$ is a pair, consisting of some set V, the so-called *vertex set*, and some subset E of the set of all 2-element subsets of V, the *edge set*. If a path starts at one vertex and ends at a different vertex, then it is called an *open path*.

A graph G is *planar* if it has a drawing without crossings. A positive integer r such that any plane embedding of a planar graph G has at least r vertices in the interior region of G is called the *inner vertex number* of G and is denoted by $i(G)$. If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be *outerplanar*, i.e., $i(G) = 0$. An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing outerplanarity. A graph G is said to be *minimally nonouterplanar* if $i(G)$=1.

A *directed graph* (or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pairs of distinct vertices called *arcs*. Here $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. For an arc (u, v) or uv in D, the first vertex u is its *tail* and the second vertex v is its *head*. For an arc $e = (u, v)$, we say that u is a *neighbor* of v; and u is *adjacent* to e and e is *adjacent* to v. A vertex u is *adjacent* to v if the arc uv is in D; u is *adjacent* from v if vu is in D.

The *total degree* $td(v)$ of a vertex v is the number of arcs incident with v, that is, $td(v) = d^-(v) + d^+(v)$. A *source* is any vertex of in-degree zero and a *sink* is a vertex of out-degree zero. By a *subgraph* of a digraph D we mean a digraph whose vertices and arcs are vertices and arcs of D. A subgraph with a certain special property is said to be *maximal with respect to that property* if no larger subgraph (i.e., with more vertices or arcs) contains it as a subgraph and has the property. For more details on maximal with respect to other special properties, see [4]. A *block* B of a digraph D is a maximal weak subgraph of D, which has no vertex v such that $B - v$ is disconnected. The following characterization of blocks are well known.

Theorem 2.1. (**Harary** [4]) : Two distinct blocks of a digraph have at most one vertex and no arcs in common.

Theorem 2.2. (**Harary** [4]) : Every arc of a digraph lies in exactly one block.

Digraphs that can be drawn without crossings between arcs (except at end vertices) are called *planar digraphs*. Clearly this property does not depend on the orientation of the arcs and hence we ignore the orientation while defining the planarity; outerplanarity; maximal outerplanarity; and minimally nonouterplanarity of a digraph.

Since most of the results and definitions of undirected graphs are valid for planar digraphs as far as their underlying graphs are concerned, the following definitions hold good for planar digraphs. A digraph D is said to be *outerplanar* if $i(D) = 0$ and *minimally nonouterplanar* if $i(D) = 1$.

3 Definition of $DPBV(A_r)$

Definition 3.1. An *arborescence* A_r is a directed graph in which, for a vertex u called the *root* and any other vertex v, there is exactly one directed path from u to v.

Furthermore, by Theorem 2.1 and 2.2, every arc of an arborescence is a block.

Definition 3.2. The *root arc* of an arborescence A_r is an arc which is directed out of the root of A_r, i.e., the *root arc* of A_r is an arc whose tail is the root of A_r.

Definition 3.3. If a directed path $\vec{P_n}$ on $n \geq 2$ vertices starts at one vertex and ends at a different vertex, then $\vec{P_n}$ is called an *open directed path*.

Definition 3.4. The *directed pathos* of an arborescence A_r is defined as a collection of minimum number of arc disjoint open directed paths whose union is A_r.

Definition 3.5. The *directed path number* k' of A_r is the number of directed paths in any directed pathos of A_r, and is equal to the number of sinks in A_r, i.e., k' equals the number of sinks in A_r.

Note that the directed path number k' of an arborescence A_r is "minimum" only when the out-degree of the root of A_r is exactly one. Therefore, unless otherwise specified, the out-degree of the root of every arborescence is exactly one. Finally, we assume that the direction of the directed pathos is along the direction of the blocks (arcs) in A_r.

Definition 3.6. For an arborescence A_r, the *block vertex digraph* $Q = BV(A_r)$ has vertex set $V(Q) = V(A_r) \cup B(A_r)$, where $B(A_r)$ is the set of blocks (arcs) of A_r, and arc set

$$A(Q) = \left\{ \begin{array}{l} (B_1, v), (v, B_2); v \in V(A_r); B_1, B_2 \in B(A_r), \text{when } v \text{ is a head of the block } B_1 \\ \text{and tail of the block } B_2 \text{ in } A_r. \end{array} \right\}$$

Definition 3.7. A *directed pathos block vertex digraph* of an arborescence A_r, written $Q = DPBV(A_r)$, is the digraph whose vertex set $V(Q) = V(A_r) \cup B(A_r) \cup P(A_r)$, where $B(A_r)$ is the set of blocks (arcs) and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: for every $v \in V(A_r)$, all arcs $(B_1, v); (v, B_2)$; for which v is a head of the block B_1 and tail of the block B_2 in A_r; (P, B) such that $B \in B(A_r)$ and $P \in P(A_r)$ and the block B lies on the directed path P; (P_i, P_j) such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j.

Note that there is freedom in marking the directed paths of a directed pathos of an arborescence A_r in different ways, provided that the directed path number k' of A_r is fixed. Since the order of marking the directed paths of a directed pathos of an arborescence is not unique, the corresponding directed pathos block vertex is also not unique. This obviously raises the question of the existence of "unique" directed pathos block vertex digraph.

One can easily check that if the directed path number of an arborescence is exactly one (i.e., $k' = 1$), then the corresponding directed pathos block vertex digraph is unique. For example, the directed path number of a directed path $\vec{P_n}$ on $n \geq 2$ vertices is exactly one (since $\vec{P_n}$ has exactly one sink). Thus the directed pathos block vertex digraph of a directed path is unique. Furthermore, one can also observe easily that, for different ways of marking of the directed paths of a directed pathos of an arborescence whose underlying graph is a star graph $K_{1,n}$ on $n \geq 3$ vertices, the corresponding directed pathos block vertex digraphs are isomorphic.

See Figure 2 and Figure 3 for an example of an arborescence (with directed pathos) and its directed pathos block vertex digraph.

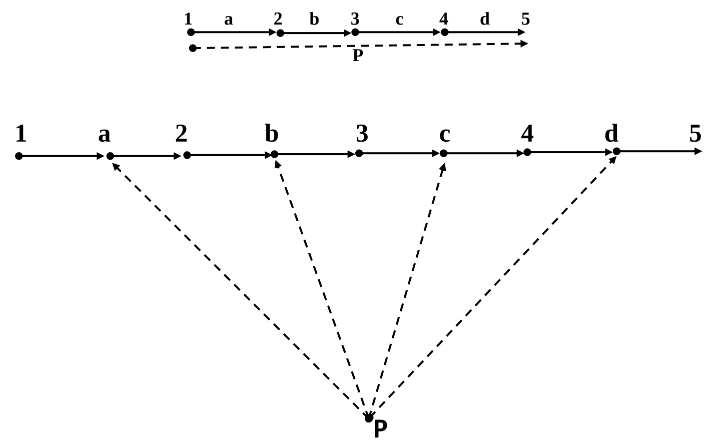

Figure 2

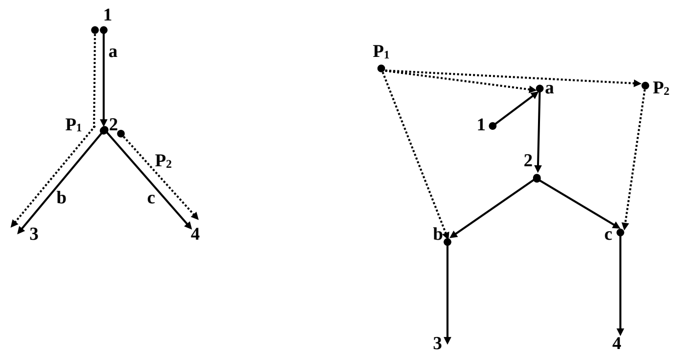

Figure 3

4 A criterion for directed pathos block vertex digraphs

The main objective is to determine a necessary and sufficient condition that a digraph be a directed pathos block vertex digraph.

A *complete bipartite digraph* is a directed graph D whose vertices can be partitioned into non-empty disjoint sets A and B such that each vertex of A has exactly one arc directed towards each vertex of B and such that D contains no other arc.

Let A_r be an arborescence with vertex set $V(A_r) = \{v_1, v_2, \ldots, v_n\}$ and a directed pathos set $P(A_r) = \{P_1, P_2, \ldots, P_t\}$. We consider the following cases.

Case 1: An arc $e = (u, v)$ with $d^+(u) = d^-(v) = 1$ give rise to a complete bipartite subdigraph with u as the tail and e head. This contributes $(n-1)$ arcs to $DPBV(A_r)$.

Case 2: An arc $e = (u, v)$ with $d^+(u) = d^-(v) = 1$ give rise to a complete bipartite subdigraph with e as the tail and v head. This also contributes $(n-1)$ arcs to $DPBV(A_r)$.

Case 3: Let P_j be a directed path which lies on r number of arcs in A_r. Then r arcs give rise to a complete bipartite subdigraph with a single tail P_j and r heads and r arcs joining P_j with each head. This again contributes $(n-1)$ arcs to $DPBV(A_r)$.

Case 4: Let P_j be a directed path and let s be the number of directed paths whose heads are reachable from the tail of P_j through a common vertex in A_r. Then s arcs give rise to a complete bipartite subdigraph with a single tail P_j and s heads and s arcs joining P_j with each head. This contributes $(k^{'} - 1)$ arcs to $DPBV(A_r)$.

Hence by all the cases above, $Q = DPBV(A_r)$ is decomposed into mutually arc-disjoint complete bipartite subdigraphs with $V(Q) = V(A_r) \cup B(A_r) \cup P(A_r)$ and arcs, (i) thrice the size of A_r, i.e., $3(n-1)$; and (ii) $k^{'} - 1$ (see Figure 4).

Conversely, let $A_r^{'}$ be a digraph of the type described above. Let $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ and $\beta_1, \beta_2, \ldots, \beta_{n-1}$ be the complete bipartite subdigraphs of Case 1 and Case 2, respectively. Let $T_i = \alpha_i \cup \beta_i$ for $1 \leq i \leq n-1$. Clearly, each T_i is a directed path $\vec{P_3}$ (see Figure 5).

Let $t_1, t_2, \ldots, t_{n-1}$ be the vertices corresponding to each T_i. Finally, let t_0 be a vertex chosen arbitrarily. For each vertex v of the complete bipartite subdigraphs T_i, we draw an arc a_v as follows:

(a) If $d^+(v) = 1$, $d^-(v) = 0$, then $a_v := (t_0, t_i)$, where i is the base (or index) of T_i such that $v \in \alpha_i$.

(b) If $d^-(v) = 1$, $d^+(v) = 1$, then $a_v := (t_i, t_j)$, where i and j are the indices of T_i and T_j such that $v \in \alpha_j \cap \beta_i$.

Let $P_1^{'}, P_2^{'}, \ldots, P_t^{'}$ be the complete bipartite subdigraphs of Case 3. We now mark the directed paths of a directed pathos as follows. It is easy to observe that the directed path number $k^{'}$ equals the number of subdigraphs of Case 3. Let $\psi_1, \psi_2, \ldots, \psi_t$ be the number of heads of subdigraphs $P_1^{'}, P_2^{'}, \ldots, P_t^{'}$, respectively. Suppose we mark the directed path P_1. For this we choose any ψ_1 number of arcs and mark P_1 on ψ_1 arcs. Similarly, we choose ψ_2 number of arcs and mark P_2 on ψ_2 arcs. This process is repeated until all the directed paths are marked. The digraph A_r with directed pathos thus constructed apparently has $A_r^{'}$ as directed pathos block vertex digraph. Thus we have,

Theorem 4.1. A digraph $A_r^{'}$ is a directed pathos block vertex digraph of an arborescence A_r if and only if $V(Q) = V(A_r) \cup B(A_r) \cup P(A_r)$ and arcs (i) thrice the size of A_r, i.e., $3(n-1)$ and (ii) $k^{'} - 1$.

Given a directed pathos block vertex digraph Q, the proof of the sufficiency of Theorem 4.1 shows how to find an arborescence A_r such that $DPBV(A_r) = Q$. This obviously raises the question of whether Q determines A_r "uniquely". Although the answer to this in general is no, the extent to which A_r is determined is given as follows.

One can check easily that using the reconstruction procedure of the sufficiency of Theorem 4.1, any arborescence A_r (without directed pathos) is uniquely reconstructed from its directed pathos block vertex digraph, but if the directed path number is more than one (i.e., $k^{'} > 1$), then it is not possible to mark the directed paths of directed pathos of A_r uniquely. This clearly indicates the fact that $k^{'}$ must be exactly one for the unique reconstruction of A_r together with its directed pathos. It is known that a directed path is a special case of an arborescence. Since the directed path number of $\vec{P_n}$ is one, it follows that only $\vec{P_n}$ can be reconstructed uniquely from its $DPBV(\vec{P_n})$.

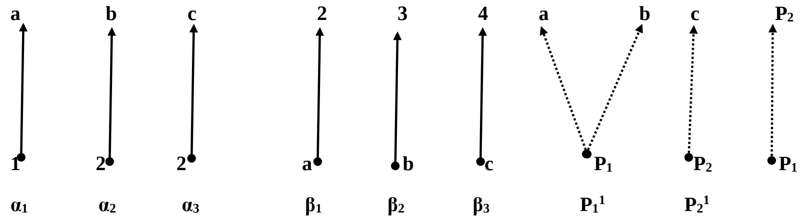

Figure 4: Decomposition of $DPBV(A_r)$ (showed in Figure 3) into complete bipartite sub digraphs.

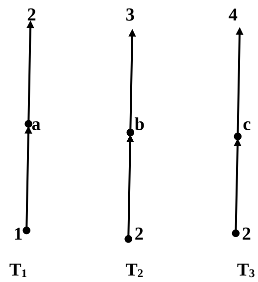

Figure 5: $T_i = \alpha_i \cup \beta_i$ for $1 \leq i \leq 3$

We now illustrate the procedure given in the converse part of Theorem 4.1. From Figure 5, $T_1 = \{1, a, 2\}$; $T_2 = \{2, b, 3\}$; and $T_3 = \{2, c, 4\}$. Thus the vertices of T_i are t_0, t_1, t_2, t_3. Considering vertices in the lexicographic order, we obtain the following arcs (in this order): $(t_0, t_1), (t_1, t_2), (t_1, t_3)$.

5 Properties of $DPBV(A_r)$

In this section we present some of the properties of $DPBV(A_r)$.

Property 5.1. For an arborescence A_r, $BV(A_r) \subseteq M(A_r)$ and $BV(A_r) \subseteq DPBV(A_r)$, where $\subseteq$ is the subdigraph notation.

Property 5.2. Let v be vertex of $DPBV(A_r)$ corresponding to the root arc of A_r. Then the removal of v from $DPBV(A_r)$ makes it a disconnected. Therefore, $DPBV(A_r)$ cannot be a block.

Property 5.3. If v is the root of A_r, then $d^-(v) = 0$ and $d^+(v) = 1$ in $DPBV(A_r)$. Therefore, $DPBV(A_r)$ is always non-Eulerian (since $d^-(v) \neq d^+(v)$).

Property 5.4. Let v be the vertex of $DPBV(A_r)$ corresponding to the root arc (a pendant arc) of A_r. Then $d^-(v) = 2$ and $d^+(v) = 1$. Thus the total degree of v is always three.

Property 5.5. The total degree of a vertex v in A_r equals the total degree of the corresponding vertex v in $DPBV(A_r)$.

In order to prove the next property, we need the following theorem and definitions.

Theorem 5.6. (**Gutin** [6]) : Let D be an acyclic digraph with precisely one source x in D. Then for every $v \in V(D)$, there is an $(x, v)-$directed path in D.

A *transmitter* is a vertex v whose out-degree is positive and whose in-degree is zero, i.e., $d^+(v) > 0$ and $d^-(v) = 0$. A *carrier* is a vertex v whose out-degree and in-degree are both one, i.e., $d^+(v) = d^-(v) = 1$. A *receiver* is a vertex v whose out-degree is zero and whose in-degree is positive, i.e., $d^+(v) = 0$ and $d^-(v) > 0$. A vertex v is said to be *ordinary* if $d^+(v) > 0$ and $d^-(v) > 0$.

Definition 5.7. A *directed pathos vertex* of $DPBV(A_r)$ is a vertex corresponding to the directed path of a directed pathos of A_r.

Proposition 5.8. Let A_r be an arborescence of order n ($n \geq 2$) with v_1 and $e_1 = (v_1, v_2)$ as the root and root arc of A_r, respectively. Then there exists exactly one vertex v with $d^+(v) > 0$ and $d^-(v) = 0$ (i.e., transmitter), and for every vertex $x \in DPBV(A_r)$ (except for the vertex v_1), there is an $(v, x)-$ directed path in $DPBV(A_r)$.

Proof: Let A_r be an arborescence with vertex set $V(A_r) = \{v_1, v_1, \ldots, v_n\}$ and arc set $A(A_r) = \{e_1, e_2, \ldots, e_{n-1}\}$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively. By definition of $L(A_r)$, the vertices $e_2, e_3, \ldots, e_{n-1}$ are reachable from e_1 by a unique directed path. Let $P(A_r) = \{P_1, P_2, \ldots, P_{k'}\}$ be a directed pathos set of A_r such that P_1 lies on the arc e_1. Since the direction of the directed pathos is along the direction of the arcs in A_r, in $DPBV(A_r)$, $d^+(v_1) = 1, d^-(v_1) = 0$; $d^+(P_1) > 0$, $d^-(P_1) = 0$; and the remaining vertices are either receiver or carrier or ordinary. Clearly, $DPBV(A_r)$ is acyclic. By Theorem 5.6, for every (except v_1) vertex $x \in DPBV(A_r)$, there is an $(P_1, x)-$ directed path in $DPBV(A_r)$. This completes the proof. ■

While defining any class of digraphs, it is desirable to know the order and size of each; it is easy to determine for $DPBV(A_r)$.

Proposition 5.9. Let A_r be an arborescence with n vertices $v_1, v_2, \ldots, v_n$ and $k^{'}$ sinks. Then the order and size of $DPBV(A_r)$ are $2n + k^{'} - 1$ and $3n + k^{'} - 4$, respectively.

Proof: If A_r has n vertices and $k^{'}$ sinks, then it follows immediately that $DPBV(A_r)$ contains $n + n - 1 + k^{'} = 2n + k^{'} - 1$ vertices. Furthermore, every arc of $DPBV(A_r)$ corresponds to an arc in A_r (there are $n - 1$ arcs); an arc adjacent to a vertex in A_r (there are $n - 1$ of these); a vertex adjacent to an arc in A_r (there are $n - 1$ of these); and the arcs whose end-vertices are the directed pathos vertices (this is given by $k^{'} - 1$). Therefore, $DPBV(A_r)$ has $(n - 1) + 2(n - 1) + k^{'} - 1 = 3n + k^{'} - 4$ arcs. ■

6 Characterization of $DPBV(A_r)$

6.1 Planar directed pathos block vertex digraphs

We characterize the digraphs whose $DPBV(A_r)$ is planar.

Theorem 6.1. A directed pathos block vertex digraph $DPBV(A_r)$ of an arborescence A_r is planar if A_r is either a directed path $\vec{P_n}$ on $n \geq 2$ vertices or the underlying graph of A_r is a star graph $K_{1,n}$ on $n \geq 3$ vertices.

Proof: Suppose that A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices. Let $V(\vec{P_n}) = \{v_1, v_2, \ldots, v_n\}$ and the arcs of $\vec{P_n}$ be $e_i = (v_i, v_{i+1})$ for $1 \leq i \leq n - 1$. By definition, $BV(A_r)$ is also a directed path of order $n + n - 1 = 2n - 1$ and arcs (v_i, e_i) and (e_i, v_{i+1}) for $1 \leq i \leq n - 1$. The directed path number of $\vec{P_n}$ is one, say P, and the corresponding directed pathos vertex P is a neighbor of every vertex (i.e., e_i for $1 \leq i \leq n - 1$) of $BV(A_r)$. This shows that the crossing number of $DPBV(A_r)$ is zero. Thus $DPBV(A_r)$ is planar.

On the other hand, suppose that the underlying graph of A_r is a star graph $K_{1,n}$ on $n \geq 3$ vertices. Let $V(A_r) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set and $A(A_r) = \{e_1, e_2, \ldots, e_{n-1}\}$ be the arc set of A_r such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively; and $e_i = (v_2, v_{i+1})$ for $2 \leq i \leq n$. By definition, the arcs of $BV(A_r)$

are (v_i, e_i) for $1 \leq i \leq 2$; (v_2, e_i) for $3 \leq i \leq n$; and (e_i, v_{i+1}) for $1 \leq i \leq n$. Let $P(A_r) = \{P_1, P_2, \ldots, P_{n-1}\}$ be a directed pathos set of A_r such that P_1 lies on the arcs e_1, e_2; and P_i lies on the arcs e_{i+1} for $2 \leq i \leq n-1$. Then the directed pathos vertex P_1 is a neighbor of the vertices e_1, e_2, P_i; and P_i is a neighbor of e_{i+1} for $2 \leq i \leq n-1$. This again shows that the crossing number of $DPBV(A_r)$ is zero. Thus $DPBV(A_r)$ is planar. ■

We now establish a characterization of digraphs whose $DPBV(A_r)$ are outerplanar; maximal outerplanar; and minimally nonouterplanar.

Theorem 6.2. A directed pathos middle digraph $DPBV(A_r)$ of an arborescence A_r is outerplanar if and only if A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices.

Proof: Suppose that $DPBV(A_r)$ is outerplanar. Assume that there exist a vertex of total degree three in A_r, i.e., the underlying graph of A_r is $K_{1,3}$. Let $V(A_r) = \{v_1, v_2, v_3, v_4\}$ and $A(A_r) = \{e_1, e_2, e_3\}$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of A_r, respectively; and $e_i = (v_2, v_{i+1})$ for $2 \leq i \leq 3$. By definition, the arcs of $BV(A_r)$ are (v_i, e_i) for $1 \leq i \leq 2$; (v_2, e_3); and (e_i, v_{i+1}) for $1 \leq i \leq 3$. Let $P(A_r) = \{P_1, P_2\}$ be a directed pathos set of A_r such that P_1 lies on the arcs e_1, e_2; and P_2 lies on the arc e_3. Then the directed pathos vertex P_1 is a neighbor of the vertices e_1, e_2, P_2; and P_2 is a neighbor of the vertex e_2. This shows that the inner vertex number of $DPBV(A_r)$ is more than one, i.e., $i(DPBV(A_r)) > 1$, a contradiction (see Figure 3).

Conversely, suppose that A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices. By Theorem 6.1, $\mathrm{cr}(DPBV(A_r)) = 0$, but $i(DPBV(A_r)) = 0$ (see Figure 2). Thus $DPBV(A_r)$ is outerplanar. ■

Theorem 6.3. For any arborescence A_r, $DPBV(A_r)$ is not maximal outerplanar.

Proof: We use contradiction. Suppose that $DPBV(A_r)$ is maximal outerplanar. Assume that A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices. By Theorem 6.2, $DPBV(A_r)$ is outerplanar. Furthermore, the addition of an arc does not alter the outerplanarity of $DPBV(A_r)$. Therefore, $DPBV(A_r)$ is not maximal outerplanar, a contradiction. This completes the proof. ■

Theorem 6.4. For any arborescence A_r, $DPBV(A_r)$ is not minimally nonouterplanar.

Proof: We use contradiction. Suppose that $DPBV(A_r)$ is minimally nonouterplanar, i.e., $i(DPBV(A_r)) = 1$. We consider the following two cases.

Case 1. Suppose that A_r is a directed path $\vec{P_n}$ on $n \geq 2$ vertices. By Theorem 6.2, $DPBV(A_r)$ is outerplanar, a contradiction.
Case 2. Suppose there exist a vertex of total degree three in A_r. By necessity of Theorem 6.2, $i(DPBV(A_r)) > 1$, again a contradiction. Hence by all the cases above, $DPBV(A_r)$ is not minimally nonouterplanar. ∎

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