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The Forcing Connected Weak Edge Detour Number of a Graph

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Abstract

In this paper, we define the forcing connected weak edge detour set (number ($fcdn_w(G)$), basis). Also, the forcing connected weak edge detour number of certain standard of graphs are determined. It is shown that for each pair a, b of integers with $0 \leq a \leq b$ and $b \geq 3$, there is a connected graph G with $fcdn_w(G) = a$ and $cdn_w(G) = b$, where $cdn_w(G)$ is connected weak edge detour number of G.

Key words: Connected weak edge detour number, forcing weak edge detour number, forcing connected weak edge detour number.

2010 Mathematics Subject Classification : 05C40, 05C69

1 Introduction

By a *graph* $G = (V, E)$, we mean a finite undirected connected simple graph. The order and size of G are denoted by n and m respectively. For basic definitions and terminologies, we refer to $[1, 4]$.

For vertices u and v in a connected graph G, the *distance* $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ *geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* (G) and the maximum eccentricity is its *diameter*, *diam* (G) of G.

For vertices u and v in a connected graph G, the *detour distance* $D(u, v)$ is the length of a longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ *detour*. The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v

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to a vertex of G. The *detour radius*, $rad_D\ (G)$ of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D(G)$ of G is the maximum detour eccentricity among the vertices of G. A vertex x is said to lie on a u–v detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour set and any detour set of order $dn(G)$ is called a *detour basis* of G. These concepts were studied by Chartrand et al.[3].

A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique minimum detour set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing detour number* of S, denoted by $fdn(S)$, is the cardinality of a minimum forcing subset for S. The forcing detour number of S, denoted by $fdn(S)$, is the cardinality of forcing subset for S. The forcing detour number of G is $fdn(G) = min\{fdn(S)\}$, where the minimum is taken over all minimum connected weak edge detour sets S in G. The forcing detour number of a graph was introduced and studied by Chartrand et.al [3].

Let G be a connected graph and S a connected detour basis of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique minimum detour set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing connected detour number* of S, denoted by $fcdn(S)$, is the cardinality of a minimum forcing subset for S. The forcing detour number of S, denoted by $fcdn(S)$, is the cardinality of forcing subset for S. The forcing detour number of G is $fcdn(G) = min\{fcdn(S)\}$, where the minimum is taken over all minimum connected detour sets S in G. The forcing connected detour number was introduced and studied by Santhakumaran and Athisayanathan [5].

Let G be a connected graph and S a connected weak detour basis of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique minimum detour set containing T. A forcing subset for S of minimum cardinality is a *minimum forcing subset* of S. The *forcing weak edge detour number* of S, denoted by $fcdn(S)$, is the cardinality of a minimum forcing subset for S. The forcing weak detour number of S, denoted by $fdn_w(S)$, is the cardinality of forcing subset for S. The forcing weak detour number of G is $fdn_w(G) = min\{fdn_w(S)\}$, where the minimum is taken over all minimum weak edge detour sets S in G. In 2010, the forcing weak edge detour number of a graph was introduced and studied by Santhakumaran and Athisayanathan [6].

A set $S \subseteq V$ is called a *connected weak edge detour set* of a graph G, if S is a weak edge detour set and the subgraph $\langle S\rangle$ induced by S is connected. The minimum cardinality of a connected weak edge detour set of G is *connected weak edge detour number*, denoted by $cdn_w(G)$ of G and any connected weak edge detour set of order $cdn_w(G)$ is called a *connected weak edge detour basis* of G. These concepts were studied by Prabakar and Athisayanathan [7].

The following theorems are used in sequel.

Theorem 1.1. [7] Let G be the complete graph K_n $(n \geq 2)$. Then a set $S \subseteq V$ is a connected weak edge detour basis of G if and only if S consists of any two vertices of G.

Theorem 1.2. [7] Let G be the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$. Then a set $S \subseteq V$ is a connected weak edge detour basis of G if and only if S consists of any two adjacent vertices of G.

Theorem 1.3. [7] Let G be a cycle of order $n \geq 3$. Then a set $S \subseteq V$ is a connected weak edge detour basis of G if and only if S consists of any two adjacent vertices of G.

Theorem 1.4. [7] For any graph G of order $n \geq 2$, $2 \leq cdn_w(G) \leq n$.

Theorem 1.5. [7] Let $G = (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $n \geq 4$ such that $r \geq 1$, each $n_i \geq 2$ and $n_1 + n_2 + \ldots + n_r + k = n - 1$. Then $cdn_w(G) = r + k + 1$.

Theorem 1.6. [7] For any connected graph G with k end-vertices and l cut-vertices, $max\{2, k + l\} \leq cdn_w(G) \leq n$.

Throughout this paper G denotes a connected graph with at least two vertices.

2 Forcing subsets in Connected Weak Edge Detour Sets

Even though every connected graph contains a connected weak edge detour basis, some connected graph may contain several connected weak edge detour bases. For each connected weak edge detour basis S in a connected graph G, there is always some subset T of S that uniquely determines S as the connected weak edge detour basis containing T. Such "*forcing subsets*" will be considered in this section.

Definition 2.1. Let G be a connected graph and S a connected weak edge detour basis of G. A subset $T \subseteq S$ is called a *forcing subset* for S if S is the unique connected weak edge detour basis containing T. A forcing subset for S of minimum cardinality is a *minimum* *forcing subset* of S. The *forcing connected weak edge detour number* of S, denoted by $fcdn_w(S)$, is the cardinality of a minimum forcing subset for S. The forcing connected weak edge detour number of G, denoted by $fcdn_w(G)$, is $fcdn_w(G) = min\{fcdn_w(S)\}$, where the minimum is taken over all minimum connected weak edge detour bases S in G.

Example 2.2. For the graph G given in Figure 2.1, $S = \{u, v, w, x, y\}$ is the unique connected weak edge detour basis of G so that $fcdn_w(G) = 0$. For the graph G given in Figure 2.2, $S_1 = \{u, s, w, t, v\}$, $S_2 = \{u, s, x, t, v\}$ and $S_3 = \{u, s, y, t, v\}$ are the three connected weak edge detour bases of G so that $fcdn_w(G) = 1$.

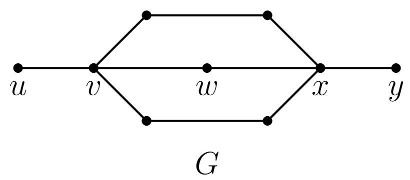


Figure 2.1

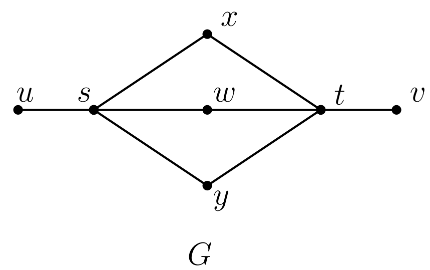


Figure 2.2

Example 2.3. For the graph G given in Figure 2.3, $fcdn(G) = 2$ and $fcdn_w(G) = 1$. Hence the forcing connected detour number and forcing connected weak edge detour number are different.

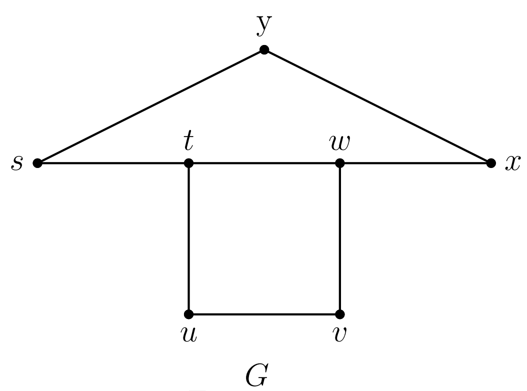


Figure 2.3

Theorem 2.4. For every connected graph G, $0 \leq fcdn_w(G) \leq cdn_w(G)$.

Proof: It is clear from the definition of $fcdn_w(G)$ that $fcdn_w(G) \geq 0$. Let S be any connected weak edge detour basis of G. Since $fcdn_w(S) \leq cdn_w(G)$ and since $fcdn_w(G) = min\{fcdn_w(S); S$ is a connected weak edge detour basis of $G\}$, it follows that $fcdn_w(G) \leq cdn_w(G)$. Thus $0 \leq fcd_n(G) \leq cdn_w(G)$. ■

Remark 2.5. The lower bound in theorem 2.4 are sharp. For the graph G given in Figure 2.1, $fcdn_w(G) = 0$. For the cycle C_3, $fcdn_w(C_3) = cdn_w(C_3) = 2$. Also, all the inequalities in theorem 2.4 can be strict. For the graph G in Figure 2.2, $fcdn_w(G) = 1$ and $cdn_w(G) = 5$ so that $0 < fcdn_w(G) < cdn_w(G)$.

In the following theorem we characterize graphs G for which the bounds in theorem 2.4 are attained and also graphs for which $fcdn_w(G) = 1$.

Theorem 2.6. Let G be a connected graph. Then

(a) $fcdn_w(G) = 0$ if and only if G has a unique connected weak edge detour basis.

(b) $fcdn_w(G) = 1$ if and only if G has at least two connected weak edge detour bases, one of which is a unique connected weak edge detour basis containing one of its elements, and

(c) $fcdn_w(G) = cdn_w(G)$ if and only if no connected weak edge detour basis of G is the unique connected weak edge detour basis containing any of its proper subsets.

Proof: (a) Let $fcdn_w(G) = 0$. Then, by definition, $fcdn_w(S) = 0$ for some connected weak edge detour basis S of G so that the empty set ϕ is the minimum forcing subset for S. Since the empty set ϕ is a subset of every set, it follows that S is the unique connected weak edge detour basis of G. The converse is clear.

(b) Let $fcdn_w(G) = 1$. Then by (a), G has at least two connected weak edge detour bases. Also, since $fcdn_w(G) = 1$, there is a singleton subset T of a connected weak edge detour basis S of G such that T is not a subset of any other connected weak edge detour basis of G. Then S is the unique connected weak edge detour basis containing one of its elements. The converse is clear.

(c) Let $fcdn_w(G) = cdn_w(G)$. Then $fcdn_w(S) = cdn_w(G)$, for every connected weak edge detour basis S in G. Also, by theorem 1.4, $cdn_w(G) \geq 2$ and hence $fcdn_w(G) \geq 2$. Then by (a), G has at least two connected weak edge detour bases and so the empty set ϕ is not a forcing subset of any weak edge detour basis of G. Since $fcdn_w(S) = cdn_w(G)$, no proper subset of S is a forcing subset of S. Thus no connected weak edge detour basis of G is the unique connected weak edge detour basis containing any of its proper subsets. Conversely, the data implies that G contains more than one connected weak edge detour basis and no subset of any connected weak edge detour basis S other than S is a forcing subset for S. Hence it follows that $fcdn_w(G) = cdn_w(G)$.

■

Theorem 2.7. Let G be a connected graph and let $\mathcal{F}$ be the set of relative complements of the minimum forcing subsets in their respective connected weak edge detour basis in G. Then $\bigcap_{F\in\mathcal{F}} F$ is the set of connected weak edge detour edges of G.

Proof: Let W be the set of all connected weak edge detour vertices of G. We claim that $W = \bigcap_{F\in\mathcal{F}} F$. Let a vertex $v \in W$. Then v is a connected weak edge detour vertex of G so that v belongs to every connected weak edge detour basis S of G. Let $T \subseteq S$ be any minimum forcing subset for any connected weak edge detour basis S of G. We claim that $v \notin T$. It $v \in T$, then $T' = T - \{v\}$ is a proper subset of T such that S is the unique connected weak edge detour basis containing T' so that T' is a forcing subset for S with $|T'| \leq |T|$, which is a contradiction to T is a minimum forcing subset for S. Thus $v \notin T$ and so $v \in F$, where F is the relative complement of T in S. Hence $v \in \bigcap_{F\in\mathcal{F}} F$ so that $W \subseteq \bigcap_{F\in\mathcal{F}} F$. Conversely, let $v \in \bigcap_{F\in\mathcal{F}} F$. Then v belongs to the relative complement of T in S for every T and every S such that $T \subseteq S$, where T is minimum forcing subset for S. Since F is the relative complement of T in S, $F \subseteq S$ and so $v \in S$, for every S so that v is a connected weak edge detour vertex of G. Thus $v \in W$ and so $\bigcap_{F\in\mathcal{F}} F \subseteq W$. Hence $W = \bigcap_{F\in\mathcal{F}} F$. ■

Corollary 2.8. Let G be a connected graph and S a connected weak edge detour basis of G. Then no connected weak edge detour vertex of G belong to minimum forcing set of G.

Proof: The proof is contained in the first part of theorem 2.7. ■

Theorem 2.9. Let G be a connected graph and W be the set of all connected weak edge detour vertices of G. Then $fcdn_w(G) \leq cdn_w(G) - |W|$.

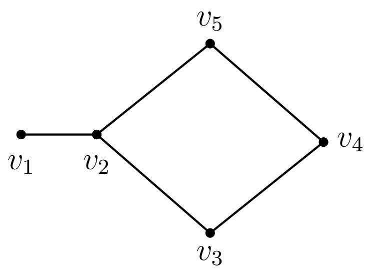
Proof: Let S be any connected weak edge detour basis of G. Then $cdn_w(G) = |S|$, $W \subseteq S$ and S is the unique connected weak edge detour basis containing $S - W$. Thus $fcdn_w(G) \leq |S - W| = |S| - |W| = cdn_w(G) - |W|$. ■

Corollary 2.10. if G is a connected graph with k end-vertices and l cut-vertices, then $fcdn_w(G) \leq cdn_w(G) - k - l$.

Proof: This follows from theorem 1.6 and theorem 2.9. ■

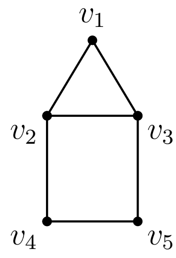
Remark 2.11. The bound in theorem 2.9 is sharp. For the graph G given in the Figure 2.4 the set $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_1, v_2, v_5\}$ are the connected detour bases of G so that $cdn_w(G) = 3$, $|W| = 2$ and $fcdn_w(G) = 1$ as in the Remark 2.5. Also, the inequality in theorem 2.9 can be strict. For the graph G given in Figure 2.5 the sets $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_1, v_2, v_4\}$ and $S_3 = \{v_1, v_3, v_5\}$ are three connected detour bases of G so that $|W| = 1$, $cdn_w(G) = 3$ and $fcdn_w(G) = 1$. Thus $fcdn_w(G) < cdn_w(G) - |W|$.

Moreover, the bound in Corollary 2.10 is also sharp. For the graph G given in Figure 2.4, $cdn_w(G) = 3$, $k = 1$, $l = 1$ and $fcdn_w(G) = 1$. Also, the inequality in Corollary 2.10 can be strict. For the graph G given in Figure 2.2, $cdn_w(G) = 5$, $k = 1$, $l = 1$ and $fcdn_w(G) = 1$. Thus $fcdn_w(G) < cdn_w(G) - k - l$.



G

Figure 2.4



G

Figure 2.5

Theorem 2.12. (a) If G is the complete graph K_n $(n \geq 3)$ or the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$, then $cdn_w(G) = fcdn_w(G) = 2$.

(b) If G is the cycle C_n $(n \geq 3)$, then $cdn_w(G) = fcdn_w(G) = 2$.

(c) If G is a tree of order $n \geq 2$, then $cdn_w(G) = fcdn_w(G) = 0$.

Proof: (a) By theorem 1.1 and theorem 1.2, a set S of vertices is a connected weak edge detour basis of G if and only if S consists of any two vertices of G. For each vertex v in G there are two or more vertices adjacent with v. Thus the vertex v belongs to more than one connected weak edge detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any connected weak edge detour basis of G. Thus the result follows.

(b) By theorems 1.3 (according as G is odd or even), a set S of two adjacent vertices of G is a connected weak edge detour basis. For each vertex v in G there are two vertices adjacent with v. Thus the vertex v belongs to more than one connected weak edge detour basis of G. Hence it follows that no set consisting of a single vertex is a forcing subset for any connected weak edge detour basis of G. Thus the result follows.

(c) By theorem 1.4, $cdn_w(G) = n$. Since the set of all vertices of a tree is the unique connected weak edge detour basis so that $fcdn_w(G) = 0$ by theorem 2.6(a).

∎

The following theorem gives a realization result.

Theorem 2.13. For each pair a, b of integers with $0 \leq a \leq b$ and $b \geq 3$, there is a connected graph G with $fcdn_w(G) = a$ and $cdn_w(G) = b$.

Proof: Case 1 : $a = 0$. For each $b \geq 3$, let G be a tree with b veritices. Then $fcdn_w(G) = 0$ and $cdn_w(G) = b$ by theorem 2.12(c).
Case 2 : $a \geq 1$. For each i with $(1 \leq i \leq a)$, let F_i be a copy of the complete graph K_2, where $V(F_i) = \{u_i, v_i\}$ and let $H = K_{1,b-a-1}$ be the star at v whose vertex set is $W = \{z_1, z_2, \ldots, z_{b-a-1}, v\}$. Then the graph G is obtained by joining the central vertex v of H to the vertices of $F_1, F_2, ..., F_a$. The graph G is connected and is shown in Figure 2.6. Then by theorem 1.5, $cdn_w(G)$=b. Now, we show that $fcdn_w(G) = a$. It is clear that W is the set of all connected weak edge detour vertices of G. Hence it follows from Therorem 2.9 that $fcdn_w(G) \leq cdn_w(G) - |W| = b - (b-a) = a$. Now, since $cdn_w(G) = b$, it follows from Therorem 1.11 that any connected weak edge detour basis of G is of the form $S = W \cup \{x_1, x_2, \ldots, x_a\}$, where $x_i \in \{u_i, v_i\}$ $(1 \leq i \leq a)$. Let T be a subset of S with $|T| < a$. Then there is a vertex x_j $(1 \leq j \leq a)$ such that $x_j \notin T$. Let y_j be a vertex of F_j distinct from x_j. Then $S' = (S - \{x_j\}) \cup \{y_j\}$ is also a connected weak edge detour basis such that it contains T. Thus S is not the unique connected weak edge detour basis containing T so T is not a forcing set of S. Since this is true for all connected weak edge detour bases of G, it follows that $fcdn_w(G) \geq a$ and so $fcdn_w(G) = a$. ∎

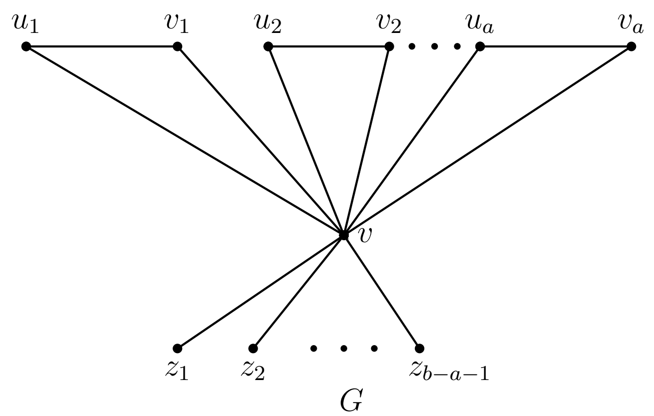


Figure 2.6

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