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General sum–connectivity coindex of graphs

M. M. Matejić
Faculty of Electronic Engineering, University of Niš
Niš, Serbia
marjan.matejic@elfak.ni.ac.rs

I. Ž. Milovanović
Faculty of Electronic Engineering, University of Niš
Niš, Serbia
igor@elfak.ni.ac.rs

E. I. Milovanović
Faculty of Electronic Engineering, University of Niš
Niš, Serbia
ema@elfak.ni.ac.rs

Abstract

Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a simple graph of order n, size m and a sequence of vertex degrees $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, $d_i = d(v_i)$. The general sum–connectivity coindex is defined as $\overline{H}_\alpha(G) = \sum_{i \nsim j}(d_i+d_j)^\alpha$, where α is an arbitrary real number and $i \nsim j$ means that vertices v_i and v_j are not adjacent. We prove a number of inequalities which determine bounds for the general sum–connectivity coindex.

Key words: Topological indices, General sum–connectivity coindex.

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1 Introduction

Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a simple graph with $n \ge 3$ vertices, m edges and a sequence of vertex degrees $d_1 \ge d_2 \ge \cdots \ge d_n > 0$, $d_i = d(v_i)$. Let $\overline{G}$ be the complement of G. If vertices v_i and v_j are adjacent in G, we write $i \sim j$. Similarly, if v_i and v_j are adjacent in $\overline{G}$, we write $i \nsim j$. The number of edges in graph $\overline{G}$ is $\overline{m} = \frac{n(n-1)}{2} - m$. We define values $\overline{\delta}_e$ and $\overline{\Delta}_e$ as

$$\overline{\delta}_e = \min_{i \nsim j}\{d_i+d_j\} \quad \text{and} \quad \overline{\Delta}_e = \max_{i \nsim j}\{d_i+d_j\}.$$

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A graph invariant is a function on a graph that does not depend on the labeling of its vertices. Such quantities are also called topological indices. The first and second Zagreb indices are vertex-degree-based graph invariants introduced in [10] and [11], respectively, and defined as

$$M_1(G)=\sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2(G)=\sum_{i\sim j} d_i d_j .$$

Both $M_1(G)$ and $M_2(G)$ were recognized to be a measure of the extent of branching of the carbon–atom skeleton of the underlying molecule.

In [10], another quantity, the sum of cubes of vertex degrees

$$F(G)=\sum_{i=1}^{n} d_i^3,$$

was encountered as well. This quantity is also a measure of branching and it was found that its predictive ability is quite similar to that of $M_1(G)$. However, for the unknown reasons, it did not attracted any attention until 2015 when it was reinvented in [9] and named the forgotten topological index.

The general sum–connectivity index, $H_\alpha(G)$, was defined in [27] as

$$H_\alpha(G)=\sum_{i\sim j}(d_i+d_j)^\alpha,$$

where α is an arbitrary real number. Obviously, for $\alpha=1$, the first Zagreb index is obtained. Another special cases include the hyper Zagreb index, $HM(G)=H_2(G)$ [22], and harmonic index, $H(G)=2H_{-1}(G)$ [7].

Most degree based topological indices are viewed as the contributions of pairs of adjacent vertices. But equally important are degree based topological indices that consider the non-adjacent pairs of vertices for computing some topological properties of graphs and named as coindices.

In [5] it was observed that the first Zagreb index can be also represented as

$$M_1(G)=\sum_{i\sim j}(d_i+d_j)\,,$$

and inspired by the above identity a concept of coindices was introduced. In this case the sum runs over the edges of the complement of G. Thus, the first and the second Zagreb coindices are defined as [5]

$$\overline{M}_1(G) = \sum_{i \nsim j} (d_i + d_j) \quad \text{and} \quad \overline{M}_2(G) = \sum_{i \nsim j} d_i d_j .$$

The forgotten topological coindex, or F-coindex, $\overline{F}(G)$, was encountered in [4] as

$$\overline{F}(G) = \sum_{i \nsim j} (d_i^2 + d_j^2).$$

The F-coindex has almost the same predictive ability for a chemically relevant property of a non-trivial class of molecules as $M_1(G)$ and $F(G)$ (see [25]). It also appears in the literature under the name Lanzhou index, see e.g. [25] and [8].

The general sum–connectivity coindex, $\overline{H}_\alpha(G)$,

$$\overline{H}_\alpha(G) = \sum_{i \nsim j} (d_i + d_j)^\alpha ,$$

where α is an arbitrary real number, was defined in [23]. Some special cases include the first Zagreb coindex, $\overline{M}_1(G) = \overline{H}_1(G)$, harmonic coindex $\overline{H}(G) = 2\overline{H}_{-1}(G)$ and the hyper–Zagreb coindex $\overline{HM}(G) = \overline{H}_2(G)$ defined in [24]. Let us note that $\overline{HM}(G)$ actually is not a new coindex. It is a linear combination of $\overline{F}(G)$ and $\overline{M}_2(G)$, that is

$$\overline{H}_2(G) = \sum_{i \nsim j} (d_i + d_j)^2 = \overline{F}(G) + 2\overline{M}_2(G).$$

The same applies for the hyper–Zagreb index.

Multiplicative versions of the first and second Zagreb indices were first considered in a paper [13] published in 2011, and were promptly followed by numerous additional studies. Multiplicative variant of the first Zagreb coindex was introduced in [26], and defined as

$$\overline{\Pi}_1(G) = \prod_{i \nsim j} (d_i + d_i).$$

More on the above mentioned indices and coindices can be found in, for example, [1, 2, 3, 12, 16, 17, 18, 19] and in the references cited therein.

In this article we prove a number of inequalities that determine upper and lower bounds for the general sum–connectivity coindex. For some particular cases of α various new/old bounds of some of the aforementioned coindices are obtained.

2 Preliminaries

In this section we recall some analytical inequalities for real number sequences that will be used later in the paper.

Let $p = (p_i)$, $i = 1, 2, \dots, n$, be a sequence of non-negative real numbers, and $a = (a_i)$, $i = 1, 2 \dots, n$, a sequence of positive real numbers. Then for any real r, $r \le 0$ or $r \ge 1$, holds [14]

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \ge \left(\sum_{i=1}^{n} p_i a_i\right)^r . \tag{1}$$

When $0 \le r \le 1$, the opposite inequality is valid. Equality holds if and only if either $a_1 = a_2 = \cdots = a_n$, or $p_1 = p_2 = \cdots = p_t = 0$ and $a_{t+1} = \cdots = a_n$, for some t, $1 \le t \le n-1$.

Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \dots, n$, be positive real number sequences. Then for any r, $r \ge 0$, holds [21]

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \ge \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^r}, \tag{2}$$

with equality if and only if $r = 0$, or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

Let $a = (a_i)$, $i = 1, 2, \dots, n$, be positive real number sequence. In [15] the following inequality was proven

$$\left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \ge \sum_{i=1}^{n} a_i + n(n-1)\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} . \tag{3}$$

Equality is attained if and only if $a_1 = a_2 = \cdots = a_n$.

3 Main results

In the next theorem we establish relationship between $\overline{H}_\alpha(G)$ and $\overline{H}_{\alpha-1}(G)$.

Theorem 3.1. Let G be a simple connected graph of size $n \ge 3$ and order m with the property $\overline{\delta}_e \ne \overline{\Delta}_e$. Then for any real α, $\alpha \le 0$ or $\alpha \ge 1$, holds

$$\overline{H}_\alpha(G) \ge \overline{\delta}_e \overline{H}_{\alpha-1}(G) + \frac{(\overline{M}_1(G) - \overline{m}\overline{\delta}_e)^\alpha}{\left(\overline{m} - \frac{1}{2}\overline{\delta}_e \overline{H}(G)\right)^{\alpha-1}} \tag{4}$$

and

$$\overline{H}_\alpha(G) \le \overline{\Delta}_e \overline{H}_{\alpha-1}(G) - \frac{(\overline{\Delta}_e \overline{m} - \overline{M}_1(G))^\alpha}{\left(\frac{1}{2}\overline{\Delta}_e \overline{H}(G) - \overline{m}\right)^{\alpha-1}}. \tag{5}$$

When $0 \le \alpha \le 1$, the opposite inequalities hold. Equalities hold if and only if either $\alpha = 0$, or $\alpha = 1$, or $d_i + d_j \in \{\overline{\delta}_e, \overline{\Delta}_e\}$, $\overline{\delta}_e \ne \overline{\Delta}_e$.

Proof: For real number α we have that

$$\overline{H}_\alpha(G) - \overline{\delta}_e \overline{H}_{\alpha-1}(G) = \sum_{i \nsim j} \left(d_i + d_j - \overline{\delta}_e\right)(d_i + d_j)^{\alpha-1}. \tag{6}$$

On the other hand, for $r = \alpha$, $\alpha \le 0$ or $\alpha \ge 1$, $p_i := \frac{d_i + d_j - \overline{\delta}_e}{d_i + d_j}$, $a_i := d_i + d_j$, with summation performed over all nonadjacent vertices in G, the inequality (1) transforms into

$$\left(\sum_{i \nsim j} \frac{d_i + d_j - \overline{\delta}_e}{d_i + d_j}\right)^{\alpha-1} \sum_{i \nsim j} (d_i + d_j - \overline{\delta}_e)(d_i + d_j)^{\alpha-1} \ge \left(\sum_{i \nsim j} \left(d_i + d_j - \overline{\delta}_e\right)\right)^\alpha,$$

that is

$$\left(\overline{m} - \frac{1}{2}\overline{\delta}_e \overline{H}(G)\right)^{\alpha-1} \left(\overline{H}_\alpha(G) - \overline{\delta}_e \overline{H}_{\alpha-1}(G)\right) \ge (\overline{M}_1(G) - \overline{\delta}_e \overline{m})^\alpha. \tag{7}$$

Equality in (7) is attained if $d_i + d_j = \overline{\delta}_e = \overline{\Delta}_e$ for every pair of nonadjacent vertices in G. Assume that this is not satisfied. In that case $\overline{m} - \frac{1}{2}\overline{\delta}_e \overline{H}(G) \ne 0$, and from the above it follows

$$\overline{H}_\alpha(G) - \overline{\delta}_e \overline{H}_{\alpha-1}(G) \ge \frac{(\overline{M}_1(G) - \overline{\delta}_e \overline{m})^\alpha}{\left(\overline{m} - \frac{1}{2}\overline{\delta}_e \overline{H}(G)\right)^{\alpha-1}},$$

from which we arrive at (4).

Equality in (7) holds if and only if either $\alpha = 0$, or $\alpha = 1$, or $d_i + d_j = \overline{\delta}_e = \overline{\Delta}_e$ for every pair of nonadjacent vertices in G, or $d_i + d_j \in \{\overline{\delta}_e, \overline{\Delta}_e\}$, $\overline{\delta}_e \ne \overline{\Delta}_e$, for every pair of nonadjacent vertices in G. Therefore, equality in (4) holds if and if $\alpha = 0$, or $\alpha = 1$, or $d_i + d_j \in \{\overline{\delta}_e, \overline{\Delta}_e\}$, $\overline{\delta}_e \ne \overline{\Delta}_e$, for every pair of nonadjacent vertices in G.

Similarly, for real number α we have that

$$\overline{\Delta}_e \overline{H}_{\alpha-1}(G) - \overline{H}_\alpha(G) = \sum_{i \nsim j} \left(\overline{\Delta}_e - d_i - d_j\right)(d_i + d_j)^{\alpha-1}. \tag{8}$$

For $r = \alpha$, $\alpha \le 0$ or $\alpha \ge 1$, $p_i := \frac{\overline{\Delta}_e - d_i - d_j}{d_i + d_j}$, $a_i := d_i + d_j$, with summation performed over all nonadjacent pairs of vertices in G, the inequality (1) becomes

$$\left(\sum_{i\not\sim j}\frac{\overline{\Delta}_e-d_i-d_j}{d_i+d_j}\right)^{\alpha-1}\sum_{i\not\sim j}(\overline{\Delta}_e-d_i-d_j)(d_i+d_j)^{\alpha-1}\geq\left(\sum_{i\not\sim j}\left(\overline{\Delta}_e-d_i-d_j\right)\right)^{\alpha},$$

That is,

$$\left(\frac{1}{2}\overline{\Delta}_e\overline{H}(G)-\overline{m}\right)^{\alpha-1}\sum_{i\not\sim j}(\overline{\Delta}_e-d_i-d_j)(d_i+d_j)^{\alpha-1}\geq(\overline{\Delta}_e\overline{m}-\overline{M}_1(G))^{\alpha}. \tag{9}$$

If $d_i+d_j=\overline{\delta}_e=\overline{\Delta}_e$ for every pair of nonadjacent vertices in G, then in (9) equality holds. Assume that this is not true. Then, from (9) we have that

$$\sum_{i\not\sim j}(\overline{\Delta}_e-d_i-d_j)(d_i+d_j)^{\alpha-1}\geq\frac{(\overline{\Delta}_e\overline{m}-\overline{M}_1(G))^{\alpha}}{\left(\frac{1}{2}\overline{\Delta}_e\overline{H}(G)-\overline{m}\right)^{\alpha-1}},$$

from which we get (5).

Equality in (9) holds if and only if either $\alpha=0$, or $\alpha=1$, or $d_i+d_j=\overline{\delta}_e=\overline{\Delta}_e$ for every pair of nonadjacent vertices in G, or $d_i+d_j\in\{\overline{\delta}_e,\overline{\Delta}_e\}$, $\overline{\delta}_e\neq\overline{\Delta}_e$, for every pair of nonadjacent vertices in G. This implies that equality in (5) holds if and only if either $\alpha=0$, or $\alpha=1$, or $d_i+d_j\in\{\overline{\delta}_e,\overline{\Delta}_e\}$, $\overline{\delta}_e\neq\overline{\Delta}_e$, for every pair of nonadjacent vertices in G.

By a similar procedure it can be proved that when $0\leq\alpha\leq 1$, the opposite inequalities are valid in (4) and (5). ∎

Corollary 3.2. Let G, $G\ncong K_n$, be a simple graph of size $n\geq 3$ and order m. Then, for any real $\alpha\geq 1$ we have

$$\overline{H}_{\alpha}(G)\geq\frac{\overline{M}_1(G)^{\alpha}}{\overline{m}^{\alpha-1}}+\overline{\delta}_e\left(\overline{H}_{\alpha-1}(G)-\frac{\overline{M}_1(G)^{\alpha-1}}{\overline{m}^{\alpha-2}}\right). \tag{10}$$

Equality holds if and only if $\alpha=1$, or d_i+d_j is constant for every pair of nonadjacent vertices in G.

Proof: According to the arithmetic-harmonic mean inequality for real number sequences (see, for example, [20]), we have that

$$\frac{1}{2}\overline{H}(G)\overline{M}_1(G)\geq\overline{m}^2.$$

From this and inequality (4) we get

$$\overline{H}_\alpha(G) \geq \overline{\delta}_e \overline{H}_{\alpha-1}(G) + \frac{(\overline{M}_1(G) - \overline{m}\overline{\delta}_e)^\alpha}{\left(\overline{m} - \overline{\delta}_e \frac{\overline{m}^2}{\overline{M}_1(G)}\right)^{\alpha-1}},$$

from which we arrive at (10). ∎

Remark 3.3. For $\alpha \geq 1$ the inequality (10) is stronger than

$$\overline{H}_\alpha(G) \geq \frac{\overline{M}_1(G)^\alpha}{\overline{m}^{\alpha-1}},$$

which is proven in [23].

Corollary 3.4. Let G, $G \ncong K_n$, be a simple graph of size $n \geq 3$ and order m. Then for any real $\alpha \geq 2$ holds

$$\overline{H}_\alpha(G) \geq \frac{\overline{M}_1(G)^2\left(\overline{M}_1(G)^{\alpha-2} - (\overline{m}\overline{\delta}_e)^{\alpha-2}\right)}{\overline{m}^{\alpha-1}} + \overline{\delta}_e^{\alpha-2}\left(\overline{F}(G) + 2\overline{M}_2(G)\right). \tag{11}$$

Equality holds if and only if $\alpha = 2$, or $d_i + d_j$ is constant for every pair of nonadjacent vertices in G.

Proof: The inequality (10) can be considered as

$$\overline{H}_\alpha(G) - \frac{\overline{M}_1(G)^\alpha}{\overline{m}^{\alpha-1}} \geq \overline{\delta}_e\left(\overline{H}_{\alpha-1}(G) - \frac{\overline{M}_1(G)^{\alpha-1}}{\overline{m}^{\alpha-2}}\right).$$

Iterating this inequality over α, it follows

$$\overline{H}_\alpha(G) - \frac{\overline{M}_1(G)^\alpha}{\overline{m}^{\alpha-1}} \geq \overline{\delta}_e^{\alpha-2}\left(\overline{H}_2(G) - \frac{\overline{M}_1(G)^2}{\overline{m}}\right),$$

from which (11) is obtained. ∎

Since $\overline{M}_1(G) \geq \overline{\delta}_e\overline{m}$, we get the following result.

Corollary 3.5. Let G be a simple graph of size $n \geq 3$ and order m. Then for any real $\alpha \geq 2$ holds

$$\overline{H}_\alpha(G) \geq \overline{\delta}_e^{\alpha-2}\left(\overline{F}(G) + 2\overline{M}_2(G)\right).$$

Equality holds if and only if $\alpha = 2$, or $d_i + d_j$ is constant for every pair of nonadjacent vertices in G.

For some specific values of parameter α, the following results are obtained.

Corollary 3.6. Let G be a simple connected graph of size $n \geq 3$ and order m with the property $\overline{\delta}_e \neq \overline{\Delta}_e$. Then

$$\overline{F}(G) \geq \overline{\delta}_e \overline{M}_1(G) + \frac{2\left(\overline{M}_1(G) - \overline{m}\overline{\delta}_e\right)^2}{2\overline{m} - \overline{\delta}_e \overline{H}(G)} - 2\overline{M}_2(G)$$

and

$$\overline{F}(G) \leq \overline{\Delta}_e \overline{M}_1(G) - \frac{2\left(\overline{\Delta}_e \overline{m} - \overline{M}_1(G)\right)^2}{\overline{\Delta}_e \overline{H}(G) - 2\overline{m}} - 2\overline{M}_2(G).$$

Equalities hold if and only if $d_i + d_j \in \{\overline{\delta}_e, \overline{\Delta}_e\}$, $\overline{\delta}_e \neq \overline{\Delta}_e$, for every pair of nonadjacent vertices in G.

Corollary 3.7. Let G be a simple connected graph of size $n \geq 3$ and order m with the property $\overline{\delta}_e \neq \overline{\Delta}_e$. Then

$$\overline{F}(G) \geq \frac{1}{2}\overline{\delta}_e \overline{M}_1(G) + \frac{\left(\overline{M}_1(G) - \overline{m}\overline{\delta}_e\right)^2}{2\overline{m} - \overline{\delta}_e \overline{H}(G)}$$

and

$$\overline{M}_2(G) \leq \frac{1}{4}\overline{\Delta}_e \overline{M}_1(G) - \frac{\left(\overline{\Delta}_e \overline{m} - \overline{M}_1(G)\right)^2}{2\left(\overline{\Delta}_e \overline{H}(G) - 2\overline{m}\right)}.$$

Equalities hold if and only if $d_i = d_j$ for every pair of nonadjacent vertices in G.

Corollary 3.8. Let G be a simple graph of size $n \geq 3$ and order m. Then

$$\overline{F}(G) \geq \frac{\overline{M}_1(G)^2}{\overline{m}} - 2\overline{M}_2(G) \tag{12}$$

and

$$\overline{F}(G) \geq \frac{\overline{M}_1(G)^2}{2\overline{m}}. \tag{13}$$

Equality in (12) holds if and only if $d_i + d_j$ is a constant for every pair of nonadjacent vertices in G. Equality in (13) holds if and only if $d_i = d_j$ for every pair of nonadjacent vertices in G.

Remark 3.9. The inequality (12) was proven in [8].

In the next theorem we establish a relation between $\overline{H}_\alpha(G)$, $\overline{H}_{\alpha-1}(G)$, $\overline{M}_1(G)$ and $\overline{\Pi}_1(G)$.

Theorem 3.10. Let G, $G \ncong K_n$, be a simple connected graph of size $n \geq 3$ and order m. Then for any real α holds

$$\overline{H}_\alpha(G) \leq \overline{M}_1(G)\overline{H}_{\alpha-1}(G) - \overline{m}(\overline{m}-1)\overline{\Pi}_1(G)^{\alpha/\overline{m}}, \tag{14}$$

and

$$\overline{H}_\alpha(G) \geq \frac{2\left(\overline{H}_{\alpha-1}(G) + \overline{m}(\overline{m}-1)\overline{\Pi}_1(G)^{(\alpha-1)/\overline{m}}\right)}{\overline{H}(G)}. \tag{15}$$

Equalities hold if and only if $d_i + d_j$ is a constant for every pair of nonadjacent vertices in G.

Proof: For any real number α we have that

$$\overline{H}_{\alpha-1}(G) = \sum_{i \nsim j} \frac{(d_i+d_j)^\alpha}{d_i+d_j} = \sum_{i \nsim j} \frac{\left((d_i+d_j)^{\frac{\alpha}{2}}\right)^2}{d_i+d_j}.$$

On the other hand, for $r=1$, $x_i := (d_i+d_j)^{\frac{\alpha}{2}}$, $a_i := d_i+d_j$, with summation performed over all pairs of nonadjacent vertices in G, the inequality (2) becomes

$$\sum_{i \nsim j} \frac{\left((d_i+d_j)^{\frac{\alpha}{2}}\right)^2}{d_i+d_j} \geq \frac{\left(\sum_{i \nsim j}(d_i+d_j)^{\frac{\alpha}{2}}\right)^2}{\sum_{i \nsim j}(d_i+d_j)},$$

that is

$$\overline{H}_{\alpha-1}(G)\overline{M}_1(G) \geq \left(\sum_{i \nsim j}(d_i+d_j)^{\frac{\alpha}{2}}\right)^2. \tag{16}$$

Further, for $a_i := (d_i+d_j)^\alpha$, with summation performed over all pairs nonadjacent vertices in G, the inequality (3) becomes

$$\left(\sum_{i \nsim j}(d_i+d_j)^{\frac{\alpha}{2}}\right)^2 \geq \sum_{i \nsim j}(d_i+d_j)^\alpha + \overline{m}(\overline{m}-1)\left(\prod_{i \nsim j}(d_i+d_j)^\alpha\right)^{1/\overline{m}},$$

That is,

$$\left(\sum_{i\nsim j}(d_i+d_j)^{\frac{\alpha}{2}}\right)^2 \geq \overline{H}_\alpha(G)+\overline{m}(\overline{m}-1)\overline{\Pi}_1(G)^{\alpha/\overline{m}}. \tag{17}$$

Combining (16) and (17) gives

$$\overline{M}_1(G)\overline{H}_{\alpha-1}(G) \geq \overline{H}_\alpha(G)+\overline{m}(\overline{m}-1)\overline{\Pi}_1(G)^{\alpha/\overline{m}},$$

from which we arrive at (14).

Equalities in (16) and (17), and therefore in (14), hold if and only if d_i+d_j is a constant for every pair of nonadjacent vertices in G.

Now, let us prove (15). For any real α we have that

$$\overline{H}_\alpha(G)=\sum_{i\nsim j}(d_i+d_j)^\alpha=\sum_{i\nsim j}\frac{(d_i+d_j)^{\alpha-1}}{\frac{1}{d_i+d_j}}=\sum_{i\nsim j}\frac{\left((d_i+d_j)^{\frac{\alpha-1}{2}}\right)^2}{\frac{1}{d_i+d_j}}. \tag{18}$$

On the other hand, for $r=1$, $x_i:=(d_i+d_j)^{\frac{\alpha-1}{2}}$, $a_i:=\frac{1}{d_i+d_j}$, with summation in (2) performed over all pairs of nonadjacent vertices in G, the inequality (2) becomes

$$\sum_{i\nsim j}\frac{\left((d_i+d_j)^{\frac{\alpha-1}{2}}\right)^2}{\frac{1}{d_i+d_j}} \geq \frac{\left(\sum_{i\nsim j}(d_i+d_j)^{\frac{\alpha-1}{2}}\right)^2}{\sum_{i\nsim j}\frac{1}{d_i+d_j}}. \tag{19}$$

Combining (18) and (19) gives

$$\overline{H}_\alpha(G) \geq \frac{2\left(\sum_{i\nsim j}(d_i+d_j)^{\frac{\alpha-1}{2}}\right)^2}{\overline{H}(G)}. \tag{20}$$

Now, for $a_i:=(d_i+d_j)^{\alpha-1}$, with summation in (3) performed over all pairs of nonadjacent vertices in G, the inequality (3) transforms into

$$\left(\sum_{i\nsim j}(d_i+d_j)^{\frac{\alpha-1}{2}}\right)^2 \geq \sum_{i\nsim j}(d_i+d_j)^{\alpha-1}+\overline{m}(\overline{m}-1)\left(\prod_{i\nsim j}(d_i+d_j)^{\alpha-1}\right)^{1/\overline{m}},$$

that is

$$\left(\sum_{i\nsim j}(d_i+d_j)^{\frac{\alpha-1}{2}}\right)^2 \geq \overline{H}_{\alpha-1}(G)+\overline{m}(\overline{m}-1)\overline{\Pi}_1(G)^{(\alpha-1)/\overline{m}}. \tag{21}$$

Now, from the above and inequality (20) we obtain (15).

Equalities in (20) and (21), and consequently in (15), hold if and only if d_i+d_j is constant for every pair of nonadjacent vertices in G. ∎

Corollary 3.11. Let G, $G \ncong K_n$, be a simple graph of size $n \geq 3$ and order m. Then

$$\overline{F}(G) \leq \overline{M}_1(G)^2 - \overline{m}(\overline{m}-1)\overline{\Pi}_1(G)^{2/\overline{m}} - 2\overline{M}_2(G) \tag{22}$$

and

$$\overline{M}_1(G) \geq \overline{m}\,\overline{\Pi}_1(G)^{1/\overline{m}}. \tag{23}$$

Equalities hold if and only if d_i+d_j is a constant for every pair of nonadjacent vertices in G.

In the next theorem we establish a lower bound for $\overline{H}_\alpha(G)$ in terms of $\overline{H}(G)$ and parameter $\overline{m}$.

Theorem 3.12. Let G, $G \ncong K_n$, be a simple connected graph of size $n \geq 3$ and order m. Then, for any real number α, $\alpha \leq -1$ or $\alpha \geq 0$, we have

$$\overline{H}_\alpha(G)\overline{H}(G)^\alpha \geq 2^\alpha \overline{m}^{\alpha+1}. \tag{24}$$

When $-1 \leq \alpha \leq 0$, the opposite inequality holds. Equality holds if and only if $\alpha=-1$, or $\alpha=0$, or d_i+d_j is a constant for every pair of nonadjacent vertices in G.

Proof: For $r=\alpha+1$, $\alpha \leq -1$ or $\alpha \geq 0$, $p_i=\frac{1}{d_i+d_j}$, $a_i=d_i+d_j$, $G \ncong K_n$, with summation performed over all nonadjacent pairs of vertices in G, the inequality (1) becomes

$$\left(\sum_{i\nsim j}\frac{1}{d_i+d_j}\right)^\alpha \sum_{i\nsim j}(d_i+d_j)^\alpha \geq \left(\sum_{i\nsim j}1\right)^{\alpha+1},$$

That is,

$$\overline{H}_\alpha(G)\overline{H}_{-1}(G)^\alpha \geq \overline{m}^{\alpha+1}. \tag{25}$$

Since $\overline{H}_{-1}(G) = \frac{1}{2}\overline{H}(G)$, from the above inequality we obtain (24).

By a similar procedure it can be proved that the opposite inequality is valid in (24) when $-1 \leq \alpha \leq 0$.

Equality in (25), that is in (24), holds if and only if $\alpha = -1$, or $\alpha = 0$, or $d_i + d_j$ is a constant for every pair of nonadjacent vertices in G. ∎

Let $\overline{SC}(G) = \overline{H}_{-1/2}(G)$ be the sum–connectivity coindex. Now we have the following corollary of Theorem 3.12.

Corollary 3.13. Let G be a simple connected graph with $m \geq 2$ edges. Then

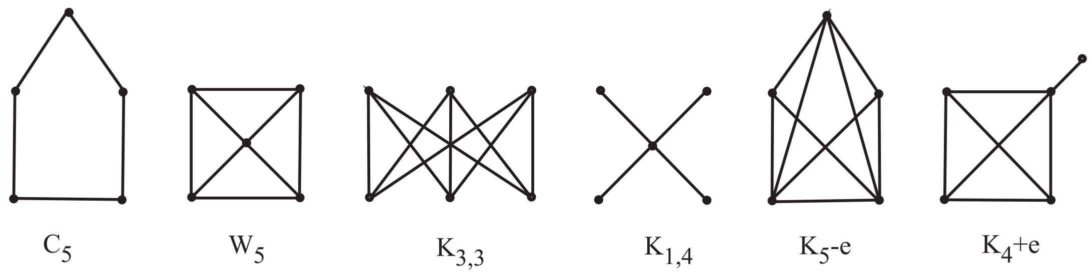
$$\overline{SC}(G) \leq \sqrt{\frac{\overline{m}\overline{H}(G)}{2}}$$

and

$$(\overline{F}(G) + 2\overline{M}_2(G))\overline{H}(G)^2 \geq 4\overline{m}^3.$$

Equality holds if and only if $d_i + d_j$ is a constant for every pair of nonadjacent vertices in G.

Remark 3.14. Let us note that there are a number of classes of connected graphs for which $d_i + d_j$ is constant for all pairs of nonadjacent vertices. Some of them are C_n (circular graph), W_n (wheel graph), $K_{\frac{n}{2},\frac{n}{2}}$ (complete bipartite graph), $K_{1,n-1}$ (star graph), $K_n - e$, $K_{n-1} + e$, etc. The following figure depicts examples of these classes for some particular values of n.



In the next theorem we prove an inequality involving relation between $\overline{H}_{\alpha+1}(G)$, $\overline{H}_{\alpha}(G)$ and $\overline{H}_{\alpha-1}(G)$, where α is an arbitrary real number.

Theorem 3.15. Let G, $G \ncong K_n$, be a connected graph with $n \geq 3$ vertices. Then for any real α holds

$$\overline{H}_{\alpha+1}(G) + \overline{\Delta}_e\overline{\delta}_e\overline{H}_{\alpha-1}(G) \leq (\overline{\Delta}_e + \overline{\delta}_e)\overline{H}_{\alpha}(G). \tag{26}$$

Equality holds if and only if $d_i + d_j \in \{\overline{\Delta}_e, \overline{\delta}_e\}$ for any two nonadjacent vertices v_i and v_j in G.

Proof: For any two nonadjacent vertices v_i and v_j in G we have that

$$(\overline{\Delta}_e - d_i - d_j)(\overline{\delta}_e - d_i - d_j) \leq 0,$$

That is,

$$(d_i + d_j)^2 + \overline{\Delta}_e\overline{\delta}_e \leq (\overline{\Delta}_e + \overline{\delta}_e)(d_i + d_j). \tag{27}$$

After multiplying the above inequality by $(d_i + d_j)^{\alpha-1}$, where α is an arbitrary real number, and summing over all nonadjacent vertices in G, we arrive at (26).

Equality in (27), and therefore in (26) is attained if and only if $d_i + d_j \in \{\overline{\Delta}_e, \overline{\delta}_e\}$ for any two nonadjacent vertices v_i and v_j in G. ∎

Corollary 3.16. Let G be a connected graph with $n \geq 2$ vertices and m edges. Then

$$\overline{HM}(G) \leq (\overline{\Delta}_e + \overline{\delta}_e)\overline{M}_1(G) - \overline{m}\overline{\Delta}_e\overline{\delta}_e.$$

Equality holds if and only if $d_i + d_j \in \{\overline{\Delta}_e, \overline{\delta}_e\}$ for any two nonadjacent vertices v_i and v_j in G.

Corollary 3.17. Let G, $G \ncong K_n$, be a connected graph with $n \geq 3$ vertices. Then

$$\overline{M}_1(G) \leq \overline{m}(\overline{\Delta}_e + \overline{\delta}_e) - \frac{1}{2}\overline{\Delta}_e\overline{\delta}_e\overline{H}(G).$$

Equality holds if and only if $d_i + d_j \in \{\overline{\Delta}_e, \overline{\delta}_e\}$ for any two nonadjacent vertices v_i and v_j in G.

Corollary 3.18. Let G, $G \ncong K_n$, be a connected graph with $n \geq 3$ vertices. Then for any real α holds

$$\overline{H}_{\alpha+1}(G) \leq \frac{(\overline{\Delta}_e + \overline{\delta}_e)^2\overline{H}_\alpha(G)^2}{4\overline{\Delta}_e\overline{\delta}_e\overline{H}_{\alpha-1}(G)}. \tag{28}$$

Equality holds if and only if $d_i + d_j$ is a constant for any two nonadjacent vertices v_i and v_j in G.

Proof: By arithmetic–geometric mean inequality, AM–GM (see e.g. [20]), based on (26) we have

$$2\sqrt{\overline{\Delta}_e\overline{\delta}_e\overline{H}_{\alpha+1}(G)\overline{H}_{\alpha-1}(G)} \leq \overline{H}_{\alpha+1}(G) + \overline{\Delta}_e\overline{\delta}_e\overline{H}_{\alpha-1}(G) \leq (\overline{\Delta}_e + \overline{\delta}_e)\overline{H}_{\alpha}(G),$$

that is

$$4\overline{\Delta}_e\overline{\delta}_e\overline{H}_{\alpha+1}(G)\overline{H}_{\alpha-1}(G) \leq (\overline{\Delta}_e + \overline{\delta}_e)^2\overline{H}_{\alpha}(G)^2,$$

from which we get (28). ∎

Corollary 3.19. Let G, $G \ncong K_n$, be a connected graph with $n \geq 3$ vertices and m edges. Then

$$\overline{F}(G) \leq \frac{(\overline{\Delta}_e + \overline{\delta}_e)^2\overline{M}_1(G)^2}{4\overline{m}\overline{\Delta}_e\overline{\delta}_e} - \overline{M}_2(G).$$

Equality holds if and only if $d_i + d_j$ is a constant for any two nonadjacent vertices v_i and v_j in G.

Corollary 3.20. Let G, $G \ncong K_n$, be a connected graph with $n \geq 3$ vertices and m edges. Then

$$\overline{M}_1(G) \leq \frac{\overline{m}^2(\overline{\Delta}_e + \overline{\delta}_e)^2}{2\overline{\Delta}_e\overline{\delta}_e\overline{H}(G)}.$$

Equality holds if and only if $d_i + d_j$ is a constant for any two nonadjacent vertices v_i and v_j in G.

Theorem 3.21. Let G, $G \ncong K_n$, be a connected graph with $n \geq 3$ vertices. Then for any real α holds

$$\overline{H}_{2\alpha+1}(G)\overline{H}(G) \geq 2\overline{H}_{\alpha}(G)^2. \tag{29}$$

Equality holds if and only if $\alpha = -1$ or $d_i + d_j$ is a constant for any two nonadjacent vertices v_i and v_j in G.

Proof: The following is valid

$$\overline{H}_{2\alpha+1}(G) = \sum_{i \nsim j}(d_i + d_j)^{2\alpha+1} = \sum_{i \nsim j}\frac{((d_i + d_j)^{\alpha})^2}{\frac{1}{d_i+d_j}}. \tag{30}$$

On the other hand, for $r = 1$, $x_i := (d_i + d_j)^{\alpha}$, $a_i := \frac{1}{d_i+d_j}$, with summation performed over all nonadjacent vertices in graph G, the inequality (2) becomes

$$\sum_{i \nsim j} \frac{((d_i+d_j)^{\alpha})^2}{\frac{1}{d_i+d_j}} \geq \frac{\left(\sum\limits_{i \nsim j}(d_i+d_j)^{\alpha}\right)^2}{\sum\limits_{i \nsim j} \frac{1}{d_i+d_j}},$$

that is

$$\sum_{i \nsim j} \frac{((d_i+d_j)^{\alpha})^2}{\frac{1}{d_i+d_j}} \geq \frac{2\overline{H}_{\alpha}(G)^2}{\overline{H}(G)}. \tag{31}$$

From the above and (30) we get (29).

Equality in (31) holds if and only if $(d_i+d_j)^{\alpha+1}$ is a constant for any two nonadjacent vertices v_i and v_j in G. Therefore, equality in (29) holds if and only if $\alpha = -1$ or $d_i + d_j$ is a constant for any two nonadjacent vertices v_i and v_j in G. ■

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