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Gutman Index of Some Graph Operations

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Abstract

In this paper, we provide the exact values of the Gutman index of join, composition, disjunction and symmetric difference of two graphs.

Key Words: Topological indices, vertex degree, join, disjunction, composition, symmetric difference, zagreb indices and coindices.

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1 Introduction

All graphs considered are simple and connected graphs. We denote the vertex and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. $d_G(v)$ denotes the degree of a vertex v in G. The number of elements in the vertex set of a graph G is called the order of G and is denoted by $v(G)$. The number of elements in the edge set of a graph G is called the size of G and is denoted by $e(G)$. A graph with order n and size m edges is called a (n, m)-graph. For any $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. The edge connecting the vertices u and v will be denoted by uv. The complement of a graph G is denoted by $\overline{G}$.

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A topological index of a graph is a parameter related to the graph, it does not depend on labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacological, toxicological, biological and other properties of chemical compounds [4]. Several types of such indices exist, expecially those based on vertex and edge distance. One of the most intensively studied topological indices is the Wiener index. The Wiener index [8] is one of the oldest molecular graph based structure descriptors[7]. Its chemical applications and Mathematical properties are well studied in [1].

The join of graphs G_1 and G_2 is denoted by G_1+G_2, and it is the graph with vertex set $V(G_1)\cup V(G_2)$ and the edge set $E(G_1+G_2)=E(G_1)\cup E(G_2)\cup\{u_1u_2|u_1\in V(G_1), u_2\in V(G_2)\}$. The composition of graphs G_1 and G_2 is denoted by $G_1[G_2]$, and it is the graph with vertex set $V(G_1)\times V(G_2)$, and two vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are adjacent if (u_1 is adjacent to v_1) or ($u_1=v_1$ and u_2 and v_2 are adjacent). The disjunction of graphs G_1 and G_2 is denoted by $G_1\vee G_2$, and it is the graph with vertex set $V(G_1)\times V(G_2)$ and $E(G_1\vee G_2)=\{(u_1,u_2)(v_1,v_2)|u_1v_1\in E(G_1)$ or $u_2v_2\in E(G_2)\}$. The symmetric difference of graphs G_1 and G_2 is denoted by $G_1\oplus G_2$, and it is the graph with vertex set $V(G_1)\times V(G_2)$ and edge set $E(G_1\oplus G_2)=\{(u_1,u_2)(v_1,v_2)|u_1v_1\in E(G_1)$ or $u_2v_2\in E(G_2)$ but not both $\}$.

Let G be a connected graph. The Wiener index $W(G)$ of a graph G is defined as

$$W(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u,v)=\frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v).$$

Dobrynin and Kochetova[2] and Gutman[5] independently presented a vertex -degree-Weighted version of Wiener index called degree distance or Schultz molecular topological index, which is defined for a connected graph G as

$$DD(G)=\sum_{\{u,v\}\subseteq V(G)} d_G(u,v)[d_G(u)+d_G(v)]=\frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)[d_G(u)+d_G(v)].$$

The Gutman index of a connected graph G, denoted by $Gut(G)$, is defined as

$$Gut(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)d_G(u)d_G(v) = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)d_G(u)d_G(v)$$

with the summation runs over all ordered pairs of vertices of G.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trianjestic [6]. The first Zagreb index $M_1(G)$ of a graph G is defined as

$$M_1(G) = \sum_{uv\in E(G)} [d_G(u) + d_G(v)] = \sum_{v\in V(G)} d_G^2(v).$$

The second Zagreb index $M_2(G)$ of a graph G is defined as

$$M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v).$$

The first Zagreb coindex $\overline{M}_1(G)$ of a graph G is defined as

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} [d_G(u) + d_G(v)].$$

The second Zagreb coindex $\overline{M}_2(G)$ of a graph G is defined as

$$\overline{M}_2(G) = \sum_{uv\notin E(G)} d_G(u)d_G(v).$$

The Zagreb indices are found to have applications in QSPR and QSAR studies as well, see [3].

In this paper, we present some exact expressions for the Gutman index of different graph operations join, composition, disjunction and symmetric difference of two graphs.

2 Basic Lemmas

Lemma 2.1. Let G_1 and G_2 be two simple connected graphs. The number of vertices and edges of graph G_i is denoted by n_i and e_i respectively for $i = 1, 2$. Then we have

1.

$$d_{G_1+G_2}(u,v)=\begin{cases}1, & uv\in E(G_1) \text{ or } uv\in E(G_2) \text{ or } (u\in V(G_1) \text{ and } v\in V(G_2))\\ 2, & \text{otherwise}\end{cases}$$

For a vertex u of G_1, $d_{G_1+G_2}(u)=d_{G_1}(u)+n_2$, and for a vertex v of $G_2, d_{G_1+G_2}(v)=d_{G_2}(v)+n_1$.

2.

$$d_{G_1[G_2]}((u_1,v_1),(u_2,v_2))=\begin{cases}d_{G_1}(u_1,u_2), & u_1\neq u_2\\ 1, & u_1=u_2, v_1v_2\in E(G_2)\\ 2, & \text{otherwise}\end{cases}$$

$d_{G_1[G_2]}(u,v)=n_2d_{G_1}(u)+d_{G_2}(v)$.

3.

$$d_{G_1\vee G_2}((u_1,v_1),(u_2,v_2))=\begin{cases}1, & u_1u_2\in E(G_1) \text{ or } v_1v_2\in E(G_2)\\ 2, & \text{otherwise}\end{cases}$$

$d_{G_1\vee G_2}((u,v))=n_2d_{G_1}(u)+n_1d_{G_2}(v)-d_{G_1}(u)d_{G_2}(v)$.

4.

$$d_{G_1\oplus G_2}((u_1,v_1),(u_2,v_2))=\begin{cases}1, & u_1u_2\in E(G_1) \text{ or } v_1v_2\in E(G_2) \text{ but not both}\\ 2, & \text{otherwise}\end{cases}$$

$d_{G_1\oplus G_2}((u,v))=n_2d_{G_1}(u)+n_1d_{G_2}(v)-2d_{G_1}(u)d_{G_2}(v)$.

Remark 2.2. For a graph G, let $A(G)=\{(x,y)\in V(G)\times V(G)\mid x \text{ and } y \text{ are adjacent in } G\}$ and let $B(G)=\{(x,y)\in V(G)\times V(G)\mid x \text{ and } y \text{ are not adjacent in } G\}$. For each $x\in V(G)$, $(x,x)\in B(G)$. Clearly, $A(G)\cup B(G)=V(G)\times V(G)$. Let $C(G)=\{(x,x)\mid x\in V(G)\}$ and $D(G)=B(G)-C(G)$. Clearly $B(G)=C(G)\cup D(G)$, $C(G)\cap D(G)=\emptyset$.

The summation $\sum\limits_{(x,y)\in A(G)}$ runs over the ordered pairs of $A(G)$. For simplicity, we write the summation $\sum\limits_{(x,y)\in A(G)}$ as $\sum\limits_{xy\in G}$. Similarly, we write the summation $\sum\limits_{(x,y)\in B(G)}$ as $\sum\limits_{xy\notin G}$. Also the summation $\sum\limits_{xy\in E(G)}$ runs over the edges of G. We denote the summation $\sum\limits_{x,y\in V(G)}$ by $\sum\limits_{x,y\in G}$. The summation $\sum\limits_{(x,y)\in D(G)}$ eqivalent to $\sum\limits_{xy\notin G, x\neq y}$ and similarly the summation $\sum\limits_{(x,y)\in C(G)}$ eqivalent to $\sum\limits_{xy\notin G, x=y}$.

Lemma 2.3. Let G be a graph. Then

$$\sum_{xy\in G} 1 = 2e(G)$$

Proof:

$$\sum_{xy\in G} 1 = 2\sum_{xy\in E(G)} 1 = 2e(G)$$

Lemma 2.4.

$$\sum_{xy\in G} d_G(x) = M_1(G)$$

Proof: Let $x\in V(G)$ and $t = d_G(x)$. Let $y_1, y_2, \dots y_t$ be the neighbours of x. Each orderd pair (x, y_i), $1 \le i \le t$, contributes $d_G(x)$ to the sum. Thus these orderd pairs contribute $d_G^2(x)$ to the sum. Hence

$$\sum_{xy\in G} d_G(x) = \sum_{x\in V(G)} d_G^2(x) = M_1(G)$$

Lemma 2.5.

$$\sum_{xy\in G} d_G(x)d_G(y) = 2M_2(G)$$

Proof: Clearly,

$$\sum_{xy\in G} d_G(x)d_G(y) = 2\sum_{xy\in E(G)} d_G(x)d_G(y) = 2M_2(G).$$

Lemma 2.6.

$$\sum_{xy \notin G} 1 = 2e(\overline{G}) + v(G)$$

Proof:

$$\begin{aligned}
\sum_{xy \notin G} 1 &= \sum_{(x,y) \in D(G)} 1 + \sum_{(x,x) \in C(G)} 1 \\
&= 2e(\overline{G}) + v(G)
\end{aligned}$$

Lemma 2.7.

$$\sum_{xy \notin G} d_G(x) = 2e(\overline{G})(v(G)-1) + 2e(G) - M_1(\overline{G})$$

Proof.

$$\begin{aligned}
\sum_{xy \notin G} d_G(x) &= \sum_{(x,y) \in D(G)} d_G(x) + \sum_{(x,x) \in C(G)} d_G(x) \\
&= \sum_{(x,y) \in D(G)} \left\{ v(G) - 1 - d_{\overline{G}}(x) \right\} + \sum_{(x,x) \in C(G)} d_G(x) \\
&= (v(G)-1) \sum_{(x,y) \in D(G)} 1 - \sum_{(x,y) \in D(G)} d_{\overline{G}}(x) + 2e(G) \\
&= (v(G)-1) 2e(\overline{G}) - \sum_{(x,y) \in A(\overline{G})} d^2_{\overline{G}}(x) + 2e(G) \\
&= (v(G)-1) 2e(\overline{G}) - \sum_{xy \in \overline{G}} d^2_{\overline{G}}(x) + 2e(G) \\
&= 2e(\overline{G})(v(G)-1) + 2e(G) - M_1(\overline{G})
\end{aligned}$$

Lemma 2.8.

$$\sum_{xy \notin G} d_G(x) d_G(y) = 2\overline{M}_2(G) + M_1(G)$$

Proof:

$$\sum_{xy \notin G} d_G(x) d_G(y) = \sum_{(x,y) \in D(G)} d_G(x) d_G(y) + \sum_{(x,x) \in C(G)} d_G(x) d_G(x)$$

$$
\begin{aligned}
&= 2\sum_{xy\notin E(G)} d_G(x)d_G(y) + \sum_{x\in V(G)} d_G^2(x)\\
&= 2\overline{M}_2(G) + M_1(G)
\end{aligned}
$$

Lemma 2.9.

$$\sum_{xy\notin G, x\neq y} d_G(x) = (v(G)-1)2e(G) - M_1(G)$$

Proof:

$$
\begin{aligned}
\sum_{xy\notin G, x\neq y} d_G(x) &= \sum_{x\in V(G)} [v(G)-1-d_G(x)]d_G(x)\\
&= (v(G)-1)\sum_{x\in V(G)} d_G(x) - \sum_{x\in V(G)} d_G^2(x)\\
&= (v(G)-1)2e(G) - M_1(G)
\end{aligned}
$$

Lemma 2.10.

$$\sum_{xy\notin G, x\neq y} 1 = (v(G)-1)v(G) - 2e(G)$$

Proof:

$$
\begin{aligned}
\sum_{xy\notin G, x\neq y} 1 &= \sum_{x\in V(G)} [v(G)-1-d_G(x)]\\
&= (v(G)-1)\sum_{x\in V(G)} 1 - \sum_{x\in V(G)} d_G(x)\\
&= (v(G)-1)v(G) - 2e(G)
\end{aligned}
$$

The Zagreb indices and Zagreb coindices, in particular the above Lemma, which is proved will be helpful in presenting our main results in a compact form.

3 Main results

In the following theorem, we compute the Gutman index of the join of two graphs.

Theorem 3.1. Let G_i be (n_i, m_i)- graph and let $\overline{m}_i = e(\overline{G}_i)$, where $i = 1, 2$. Then

$$\begin{aligned} 2 \times Gut(G_1 + G_2) &= 2M_2(G_1) + 2n_2M_1(G_1) + 2n_2^2m_1 + 4\overline{M}_2(G_1) + 4n_2[2\overline{m}_1(n_1 - 1) - M_1(\overline{G}_1)] \\ &+ 4\overline{m}_1n_2^2 + 8m_1m_2 + 4n_1m_1n_2 + 4n_2n_1m_2 + 2n_1^2n_2^2 + 2M_2(G_2) \\ &+ 2n_1M_1(G_2) + 2n_1^2m_2 + 4\overline{M}_2(G_2) + 4n_1[2\overline{m}_2(n_2 - 1) - M_1(\overline{G}_2)] + 4\overline{m}_2n_1^2 \end{aligned}$$

Proof:

$$\begin{aligned} 2 \times Gut(G_1 + G_2) &= \sum_{x \in V(G_1+G_2)} \sum_{y \in V(G_1+G_2)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &= \sum_{x \in V(G_1+G_2)} \Big[\sum_{y \in V(G_1)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &\quad + \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \Big] \\ &= \sum_{x \in V(G_1+G_2)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &+ \sum_{x \in V(G_1+G_2)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &= \sum_{x \in V(G_1)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &+ \sum_{x \in V(G_2)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &+ \sum_{x \in V(G_1)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &+ \sum_{x \in V(G_2)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &= \sum_{x \in V(G_1)} \sum_{y \in V(G_1)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &+ 2 \sum_{x \in V(G_1)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ &+ \sum_{x \in V(G_2)} \sum_{y \in V(G_2)} d_{G_1+G_2}(x, y) d_{G_1+G_2}(x) d_{G_1+G_2}(y) \\ 2 \times Gut(G_1 + G_2) &= S_1 + 2S_2 + S_3, \end{aligned}$$

where $S_1,\ S_2,\ S_3$ are terms of the above sums taken in order. We calculate $S_1,\ S_2,\ S_3$ separately one by one. Now

$$\begin{aligned}
S_1 &= \sum_{x\in V(G_1)} \sum_{y\in V(G_1)} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y) \\
&= \sum_{x,y\in V(G_1)} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y) \\
&= \sum_{xy\in G_1} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y) + \sum_{xy\notin G_1, x\neq y} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y) \\
&\quad + \sum_{xy\notin G_1, x=y} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y) \\
&= 1\sum_{xy\in G_1} d_{G_1+G_2}(x)d_{G_1+G_2}(y) + 2\sum_{xy\notin G_1, x\neq y} d_{G_1+G_2}(x)d_{G_1+G_2}(y) + 0 \\
S_1 &= S_{1,1} + 2S_{1,2},
\end{aligned}$$

where $S_{1,1}$ and $S_{1,2}$ are terms of the above sums taken in order, which are computed as follows:

$$\begin{aligned}
S_{1,1} &= \sum_{xy\in G_1} d_{G_1+G_2}(x)d_{G_1+G_2}(y) \\
&= \sum_{xy\in G_1} (d_{G_1}(x)+n_2)(d_{G_1}(y)+n_2) \\
&= \sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y) + n_2d_{G_1}(x) + n_2d_{G_1}(y) + n_2^2 \\
&= \sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y) + n_2\sum_{xy\in G_1} d_{G_1}(x) + n_2\sum_{xy\in G_1} d_{G_1}(y) + n_2^2\sum_{xy\in G_1} 1 \\
&= 2M_2(G_1) + n_2\sum_{x\in V(G_1)} d_{G_1}^2(x) + n_2\sum_{y\in V(G_1)} d_{G_1}^2(y) + n_2^2 2m_1 \\
&= 2M_2(G_1) + 2n_2M_1(G_1) + 2n_2^2m_1
\end{aligned}$$

$$\begin{aligned}
S_{1,2} &= \sum_{xy\notin G_1, x\neq y} d_{G_1+G_2}(x)d_{G_1+G_2}(y) \\
&= \sum_{xy\notin G_1, x\neq y} (d_{G_1}(x)+n_2)(d_{G_1}(y)+n_2) \\
&= \sum_{xy\notin G_1, x\neq y} d_{G_1}(x)d_{G_1}(y) + n_2d_{G_1}(x) + n_2d_{G_1}(y) + n_2^2 \\
&= \sum_{xy\notin G_1, x\neq y} d_{G_1}(x)d_{G_1}(y) + n_2\sum_{xy\notin G_1, x\neq y} d_{G_1}(x) + n_2\sum_{xy\notin G_1, x\neq y} d_{G_1}(y) + n_2^2\sum_{xy\notin G_1, x\neq y} 1 \\
&= 2\overline{M}_2(G_1) + 2n_2[2\overline{m}_1(n_1-1) - M_1(\overline{G}_1)] + 2\overline{m}_1n_2^2
\end{aligned}$$

$$\begin{aligned}
S_2 &= \sum_{x\in V(G_1)} \sum_{y\in V(G_2)} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y)\\
&= 1\sum_{x\in V(G_1)} \sum_{y\in V(G_2)} (d_{G_1}(x)+n_2)(d_{G_2}(y)+n_1)\\
&= \sum_{x\in V(G_1)} \sum_{y\in V(G_2)} d_{G_1}(x)d_{G_2}(y)+n_1d_{G_1}(x)+n_2d_{G_2}(y)+n_1n_2\\
&= \sum_{x\in V(G_1)} d_{G_1}(x) \sum_{y\in V(G_2)} d_{G_2}(y)+\sum_{x\in V(G_1)} n_1d_{G_1}(x)\sum_{y\in V(G_2)} 1\\
&\quad +n_2\sum_{x\in V(G_1)} 1 \sum_{y\in V(G_2)} d_{G_2}(y)+n_1n_2\sum_{x\in V(G_1)} 1\sum_{y\in V(G_2)} 1\\
&= 2m_12m_2+n_12m_1n_2+n_2n_12m_2+n_1n_2n_1n_2\\
&= 4m_1m_2+2n_1m_1n_2+2n_2n_1m_2+n_1^2n_2^2\\
S_3 &= \sum_{x\in V(G_2)} \sum_{y\in V(G_2)} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y)\\
&= \sum_{x,y\in V(G_1)} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y)\\
&= \sum_{xy\in G_2} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y)+\sum_{xy\notin G_2,x\neq y} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y)\\
&\quad +\sum_{xy\notin G_2,x=y} d_{G_1+G_2}(x,y)d_{G_1+G_2}(x)d_{G_1+G_2}(y)\\
&= 1\sum_{xy\in G_2} d_{G_1+G_2}(x)d_{G_1+G_2}(y)+2\sum_{xy\notin G_2,x\neq y} d_{G_1+G_2}(x)d_{G_1+G_2}(y)+0\\
S_3 &= S_{3,1}+2S_{3,2}
\end{aligned}$$

where $S_{3,1}$ and $S_{3,2}$ are terms of the above sums taken in order,

which are computed as follows:

$$\begin{aligned}
S_{3,1} &= \sum_{xy\in G_1} d_{G_1+G_2}(x)d_{G_1+G_2}(y)\\
&= \sum_{xy\in G_2} (d_{G_1}(x)+n_2)(d_{G_1}(y)+n_2)\\
&= \sum_{xy\in G_2} d_{G_1}(x)d_{G_1}(y)+n_2d_{G_1}(x)+n_2d_{G_1}(y)+n_2^2\\
&= \sum_{xy\in G_2} d_{G_1}(x)d_{G_1}(y)+n_2\sum_{xy\in G_2} d_{G_1}(x)+n_2\sum_{xy\in G_2} d_{G_1}(y)+n_2^2\sum_{xy\in G_2} 1\\
&= 2M_2(G_2)+n_2\sum_{x\in V(G_2)} d_{G_1}^2(x)+n_2\sum_{y\in V(G_2)} d_{G_1}^2(y)+n_2^22m_2\\
&= 2M_2(G_2)+2n_1M_1(G_2)+2n_1^2m_2
\end{aligned}$$

$$\begin{aligned}
S_{3,2} &= \sum_{xy\notin G_2, x\neq y} d_{G_1+G_2}(x)d_{G_1+G_2}(y)\\
&= \sum_{xy\notin G_2, x\neq y} (d_{G_1}(x)+n_2)(d_{G_1}(y)+n_2)\\
&= \sum_{xy\notin G_2, x\neq y} \Big[d_{G_1}(x)d_{G_1}(y)+n_2d_{G_1}(x)+n_2d_{G_1}(y)+n_2^2\Big]\\
&= \sum_{xy\notin G_2, x\neq y} d_{G_1}(x)d_{G_1}(y)+n_2\sum_{xy\notin G_2, x\neq y} d_{G_1}(x)+n_2\sum_{xy\notin G_2, x\neq y} d_{G_1}(y)\\
&\quad +n_2^2\sum_{xy\notin G_2, x\neq y} 1\\
&= 2\overline{M}_2(G_2)+2n_1[2\overline{m}_2(n_2-1)-M_1(\overline{G}_2)]+2\overline{m}_2n_1^2
\end{aligned}$$

$$\begin{aligned}
2\times Gut(G_1+G_2) &= S_1+2S_2+S_3\\
&= S_{1,1}+2S_{1,2}+2S_2+S_{3,1}+2S_{3,2}
\end{aligned}$$

By substituting $S_{1,1}, S_{1,2}, S_2, S_{3,1}$ and $S_{3,2}$, we get the desired results. ∎

4 Composition

In the following theorem, we compute the Gutman index of the composition of two graphs.

Theorem 4.1. Let G_i be a (n_i, m_i)- graph and let $\overline{m}_2 = e(\overline{G}_2)$. Then

$$\begin{aligned}
2\times GutG_1[G_2] &= 4M_2(G_2)W(G_1)+2n_2M_1(G_2)DD(G_1)+4m_2n_2^2Gut(G_1)+4n_1\overline{M}_2(G_2)\\
&+ 8n_2m_1[2(n_2-1)\overline{m}_2-M_1(\overline{G}_2)]+4n_2^2M_1(G_1)\overline{m}_2+2n_1M_2(G_2)\\
&+ 4n_2m_1M_1(G_2)+2n_2^2M_1(G_1)m_2+2W(G_1)[2\overline{M}_2(G_2)+M_1(G_2)]\\
&+ 2n_2DD(G_1)[2(n_2-1)\overline{m}_2-M_1(\overline{G}_2)+2m_2]+2n_2^2Gut(G_1)[2\overline{m}_2+n_2]
\end{aligned}$$

Proof:

$$2\times GutG_1[G_2] = \sum_{x,y\in G_1}\sum_{u,v\in G_2} d_{G_1[G_2]}\Big((x,u),(y,v)\Big)d_{G_1[G_2]}(x,u)d_{G_1[G_2]}(y,v)$$

$$= \sum_{x,y\in G_1}\Big\{\sum_{uv\in G_2}\Big((x,u),(y,v)\Big)d_{G_1[G_2]}(x,u)d_{G_1[G_2]}(y,v)$$

$$\begin{aligned}
&+ \sum_{uv\notin G_2}\Big((x,u),(y,v)\Big)d_{G_1[G_2]}(x,u)d_{G_1[G_2]}(y,v)\Big\}\\
&= \sum_{xy\in G_1, x=y}\ \sum_{uv\in G_2}\Big((x,u),(y,v)\Big)d_{G_1[G_2]}(x,u)d_{G_1[G_2]}(y,v)\\
&+ \sum_{xy\in G_1, x\neq y}\ \sum_{uv\in G_2}\Big((x,u),(y,v)\Big)d_{G_1[G_2]}(x,u)d_{G_1[G_2]}(y,v)\\
&+ \sum_{xy\in G_1, x=y}\ \sum_{uv\notin G_2}\Big((x,u),(y,v)\Big)d_{G_1[G_2]}(x,u)d_{G_1[G_2]}(y,v)\\
&+ \sum_{xy\in G_1, x\neq y}\ \sum_{uv\notin G_2}\Big((x,u),(y,v)\Big)d_{G_1[G_2]}(x,u)d_{G_1[G_2]}(y,v)\\
&= J_3+J_1+J_2+J_4,
\end{aligned}$$

where J_3, J_1, J_2, J_4 are terms of the above sums taken in order. Next we calculate J_1, J_2, J_3, J_4 separately.

$$\begin{aligned}
J_1 &= \sum_{xy\in G_1, x\neq y}\ \sum_{uv\in G_2} d_{G_1[G_2]}\Big((x,u),(y,v)\Big)d_{G_1[G_2]}(x,u)d_{G_1[G_2]}(y,v)\\
&= \sum_{xy\in G_1, x\neq y}\ \sum_{uv\in G_2} d_{G_1}(x,y)\Big[d_{G_2}(u)+d_{G_1}(x)n_2\Big]\Big[d_{G_2}(v)+d_{G_1}(y)n_2\Big]\\
&= \sum_{xy\in G_1, x\neq y}\ \sum_{uv\in G_2} d_{G_1}(x,y)\Big[d_{G_2}(u)d_{G_2}(v)+n_2d_{G_2}(u)d_{G_1}(y)+n_2d_{G_1}(x)d_{G_2}(v)\\
&+ n_2^2d_{G_1}(x)d_{G_1}(y)\Big]\\
&= \Big(\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)\Big)\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)+n_2\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)d_{G_1}(y)\Big(\sum_{uv\in G_2} d_{G_2}(u)\Big)\\
&+ n_2\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)d_{G_1}(x)\Big(\sum_{uv\in G_2} d_{G_2}(v)\Big)+n_2^2\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)d_{G_1}(x)d_{G_1}(y)\Big(\sum_{uv\in G_2}1\Big)\\
&= 2M_2(G_2)\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)+n_2M_1(G_2)\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)d_{G_1}(y)\\
&+ n_2M_1(G_2)\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)d_{G_1}(x)+2m_2n_2^2\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)d_{G_1}(x)d_{G_1}(y)\\
&= 2M_2(G_2)\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)+n_2M_1(G_2)\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)[d_{G_1}(x)+G_1(y)]\\
&+ 2m_2n_2^2\sum_{xy\in G_1, x\neq y} d_{G_1}(x,y)d_{G_1}(x)d_{G_1}(y)\\
&= 4M_2(G_2)W(G_1)+2n_2M_1(G_2)DD(G_1)+4m_2n_2^2Gut(G_1)
\end{aligned}$$

$$\begin{aligned}
J_2 &= \sum_{xy\in G_1, x=y} \sum_{uv\notin G_2} d_{G_1[G_2]}\Big((x,u),(y,v)\Big) d_{G_1[G_2]}(x,u) d_{G_1[G_2]}(y,v)\\
&= \sum_{xy\in G_1, x=y} \Bigg\{ \sum_{uv\notin G_2, u=v} d_{G_1[G_2]}\Big((x,u),(y,v)\Big) d_{G_1[G_2]}(x,u) d_{G_1[G_2]}(y,v)\\
&+ \sum_{uv\notin G_2, u\neq v} d_{G_1[G_2]}\Big((x,u),(y,v)\Big) d_{G_1[G_2]}(x,u) d_{G_1[G_2]}(y,v)\Bigg\}\\
&= \sum_{xy\in G_1, x=y} \sum_{uv\notin G_2, u\neq v} d_{G_1[G_2]}\Big((x,u),(y,v)\Big) d_{G_1[G_2]}(x,u) d_{G_1[G_2]}(y,v)\\
&= 2\sum_{xy\in G_1, x=y} \sum_{uv\notin G_2, u\neq v} \Big[d_{G_2}(u) + d_{G_1}(x)n_2\Big]\Big[d_{G_2}(v) + d_{G_1}(y)n_2\Big]\\
&= 2\sum_{xy\in G_1, x=y} \sum_{uv\notin G_2, u\neq v} \Big[d_{G_2}(u)d_{G_2}(v) + n_2 d_{G_2}(u)d_{G_1}(y) + n_2 d_{G_1}(x)d_{G_2}(v) + n_2^2 d_{G_1}(x)d_{G_1}(y)\Big]\\
&= 2\Big(\sum_{xy\in G_1, x=y} 1\Big) \sum_{uv\notin G_2, u\neq v} d_{G_2}(u)d_{G_2}(v) + 2n_2\Big(\sum_{xy\in G_1, x=y} d_{G_1}(y)\Big) \sum_{uv\notin G_2, u\neq v} d_{G_2}(u)\\
&+ 2n_2\Big(\sum_{xy\in G_1, x=y} d_{G_1}(x)\Big) \sum_{uv\notin G_2, u\neq v} d_{G_2}(v) + 2n_2^2 \sum_{xy\in G_1, x=y} d_{G_1}(x)d_{G_1}(y)\Big(\sum_{uv\notin G_2, u\neq v} 1\Big)\\
&= 2n_1 2\overline{M}_2(G_2) + 2n_2 2m_1[(n_2-1)2\overline{m}_2 - M_1(\overline{G}_2)] + 2n_2 2m_1[2(n_2-1)\overline{m}_2 - M_1\overline{G}_2]\\
&\quad +2n_2^2 M_1(G_1)2\overline{m}_2\\
&= 4n_1\overline{M}_2(G_2) + 4n_2m_1[2(n_2-1)\overline{m}_2 - M_1(\overline{G}_2)] + 4n_2m_1[2(n_2-1)\overline{m}_2 - M_1\overline{G}_2]\\
&+ 4n_2^2 M_1(G_1)2\overline{m}_2\\
J_3 &= \sum_{xy\in G_1, x=y} \sum_{uv\in G_2} d_{G_1[G_2]}\Big((x,u),(y,v)\Big) d_{G_1[G_2]}(x,u) d_{G_1[G_2]}(y,v)\\
&= \sum_{xy\in G_1, x=y} \sum_{uv\in G_2} \Big[d_{G_2}(u) + d_{G_1}(x)n_2\Big]\Big[d_{G_2}(v) + d_{G_1}(y)n_2\Big]\\
&= \sum_{xy\in G_1, x=y} \sum_{uv\in G_2} \Big[d_{G_2}(u)d_{G_2}(v) + n_2 d_{G_2}(u)d_{G_1}(y) + n_2 d_{G_1}(x)d_{G_2}(v) + n_2^2 d_{G_1}(x)d_{G_1}(y)\Big]\\
&= \Big(\sum_{xy\in G_1, x=y} 1\Big) \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v) + n_2 \sum_{xy\in G_1, x=y} d_{G_1}(y)\Big(\sum_{uv\in G_2} d_{G_2}(u)\Big)\\
&+ n_2 \sum_{xy\in G_1, x=y} d_{G_1}(x)\Big(\sum_{uv\in G_2} d_{G_2}(v)\Big) + n_2^2 \sum_{xy\in G_1, x=y} d_{G_1}(x)d_{G_1}(y)\Big(\sum_{uv\in G_2} 1\Big)\\
&= 2n_1M_2(G_2) + 2n_2m_1M_1(G_2) + 2n_2m_1M_1(G_2) + 2n_2^2M_1(G_1)m_2\\
J_4 &= \sum_{xy\in G_1, x\neq y} \sum_{uv\notin G_2} d_{G_1[G_2]}\Big((x,u),(y,v)\Big) d_{G_1[G_2]}(x,u) d_{G_1[G_2]}(y,v)\\
&= \sum_{xy\in G_1, x\neq y} \sum_{uv\notin G_2} d(x,y)\Big[d_{G_2}(u) + d_{G_1}(x)n_2\Big]\Big[d_{G_2}(v) + d_{G_1}(y)n_2\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{xy\in G_1, x\neq y} \sum_{uv\notin G_2} d(x,y)\Big[d_{G_2}(u)d_{G_2}(v) + n_2 d_{G_2}(u)d_{G_1}(y) + n_2 d_{G_1}(x)d_{G_2}(v) + n_2^2 d_{G_1}(x)d_{G_1}(y)\Big]\\
&= \Big(\sum_{xy\in G_1, x\neq y} d(x,y)\Big)\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v) + n_2\Big(\sum_{xy\in G_1, x\neq y} d(x,y)d_{G_1}(y)\Big)\sum_{uv\notin G_2} d_{G_2}(u)\\
&+ n_2 \sum_{xy\in G_1, x\neq y} d(x,y)d_{G_1}(x)\Big(\sum_{uv\notin G_2} d_{G_2}(v)\Big) + n_2^2 \sum_{xy\in G_1, x\neq y} d(x,y)d_{G_1}(x)d_{G_1}(y)\Big(\sum_{uv\notin G_2} 1\Big)\\
&= 2W(G_1)[2\overline{M}_2(G_2) + M_1(G_2)] + 2n_2 DD(G_1)[2(n_2-1)\overline{m}_2 - M_1(\overline{G}_2) + 2m_2]\\
&+ 2n_2^2 Gut(G_1)[2\overline{m}_2 + n_2]
\end{aligned}
$$

By adding J_1, J_2, J_3 and J_4, we get the desired result. ∎

5 Disjunction

In the following theorem, we compute the Gutman index of the disjunction of two graphs.

Theorem 5.1.

$$
\begin{aligned}
2\times Gut(G_1\vee G_2) &= \Big(2\overline{M}_2(G_1) + M_1(G_1)\Big)\Big[2n_2^2 m_2 - 2n_2 M_1(G_2) + 2M_2(G_2) + 2n_2^2(2\overline{m}_2 + n_2)\Big]\\
&+ \Big(2\overline{M}_2(G_2) + M_1(G_2)\Big)\Big[2n_1^2 m_1 - 2n_1 M_1(G_1) + 2M_2(G_1) + 2n_1^2(2\overline{m}_1 + n_1)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \Big(2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big)\Big[2n_1n_2M_1(G_2) - 4n_1M_2(G_2)\\
&- 4n_1\Big\{2\overline{M}_2(G_2) + M_1(G_2)\Big\}\Big] + \Big(2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big)\Big[2n_1n_2M_1(G_1)\\
&- 4n_2M_2(G_1) - 4n_2\Big\{2\overline{M}_2(G_1) + M_1(G_1)\Big\}\Big] + 2n_1^2M_2(G_2)(2\overline{m}_1 + n_1)\\
&+ 2n_2^2M_2(G_1)[2\overline{m}_2 + n_2] + 4n_2^2m_2M_2(G_1) + 2n_1n_2M_1(G_1)M_1(G_2) + 4n_1^2m_1M_2(G_2)\\
&- 4n_2M_2(G_1)M_1(G_2) - 4n_1M_1(G_1)M_2(G_2) + 4M_2(G_1)M_2(G_2)\\
&+ 4n_1n_2\Big[(n_1-1)2\overline{m}_1 - M_1(\overline{G}_1) + 2m_1\Big]\Big[(n_2-1)2\overline{m}_2 - M_1(\overline{G}_2) + 2m_2\Big]\\
&- 2n_1^3M_1(G_2) + 8n_1m_1M_1(G_2) - 2M_1(G_1)M_1(G_2) + 8n_2m_2M_1(G_1)\\
&+ 2\Big[2\overline{M}_2(G_1) + M_1(G_1)\Big]\Big[2\overline{M}_2(G_2) + M_1(G_2)\Big] - 2n_2^3M_1(G_1) - 16n_1n_2m_1m_2
\end{aligned}
$$

Proof:

$$
\begin{aligned}
2 \times Gut(G_1 \vee G_2) &= \sum_{x,y \in V(G_1)} \sum_{u,v \in V(G_2)} d_{G_1 \vee G_2}((x,u)(y,v)) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \\
&= \sum_{x,y \in V(G_1)} \Big\{ \sum_{uv \in G_2} d_{G_1 \vee G_2}\Big((x,u)(y,v)\Big) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \\
&+ \sum_{uv \notin G_2} d_{G_1 \vee G_2}\Big((x,u)(y,v)\Big) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \Big\} \\
&= \sum_{x,y \in V(G_1)} \sum_{uv \in G_2} d_{G_1 \vee G_2}\Big((x,u)(y,v)\Big) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \\
&+ \sum_{x,y \in V(G_1)} \sum_{uv \notin G_2} d_{G_1 \vee G_2}\Big((x,u)(y,v)\Big) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{xy \in G_1} \sum_{uv \in G_2} d_{G_1 \vee G_2}\Big((x,u)(y,v)\Big) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \\
&+ \sum_{xy \notin G_1} \sum_{uv \in G_2} d_{G_1 \vee G_2}\Big((x,u)(y,v)\Big) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \\
&+ \sum_{xy \in G_1} \sum_{uv \notin G_2} d_{G_1 \vee G_2}\Big((x,u)(y,v)\Big) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \\
&+ \sum_{xy \notin G_1} \sum_{uv \notin G_2} d_{G_1 \vee G_2}\Big((x,u)(y,v)\Big) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \\
&= C_3 + C_1 + C_2 + C_4, \text{ where } C_1, C_2, C_3, C_4 \text{ are terms of the above sums taken in order.}
\end{aligned}
$$

$$
\begin{aligned}
C_1 &= \sum_{xy \notin G_1} \sum_{uv \in G_2} d_{G_1 \vee G_2}\Big((x,u)(y,v)\Big) d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \\
&= \sum_{xy \notin G_1} \sum_{uv \in G_2} d_{G_1 \vee G_2}(x,u) d_{G_1 \vee G_2}(y,v) \\
&= \sum_{xy \notin G_1} \sum_{uv \in G_2} \Big[n_2 d_{G_1}(x) + n_1 d_{G_2}(u) - d_{G_1}(x) d_{G_2}(u)\Big]\Big[n_2 d_{G_1}(y) + n_1 d_{G_2}(v) - d_{G_1}(y) d_{G_2}(v)\Big] \\
&= \sum_{xy \notin G_1} \sum_{uv \in G_2} \Big[n_2^2 d_{G_1}(x) d_{G_1}(y) + n_1 n_2 d_{G_1}(x) d_{G_2}(v) - n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) \\
&+ n_1 n_2 d_{G_1}(y) d_{G_2}(u) + n_1^2 d_{G_2}(u) d_{G_2}(v) - n_1 d_{G_1}(y) d_{G_2}(u) d_{G_2}(v) - n_2 d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) \\
&- n_1 d_{G_1}(x) d_{G_2}(u) d_{G_2}(v) + d_{G_1}(x) d_{G_1}(y) d_{G_2}(u) d_{G_2}(v)\Big] \\
&= n_2^2 \sum_{xy \notin G_1} \sum_{uv \in G_2} d_{G_1}(x) d_{G_1}(y) + n_1 n_2 \sum_{xy \notin G_1} \sum_{uv \in G_2} d_{G_1}(x) d_{G_2}(v) \\
&- n_2 \sum_{xy \notin G_1} \sum_{uv \in G_2} d_{G_1}(x) d_{G_1}(y) d_{G_2}(v) + n_1 n_2 \sum_{xy \notin G_1} \sum_{uv \in G_2} d_{G_1}(y) d_{G_2}(u)
\end{aligned}
$$

$$
\begin{aligned}
&+ \; n_1^2 \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v) - n_1 \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- \; n_2 \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u) - n_1 \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)\\
&+ \; \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= \; n_2^2 \sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\in G_2} 1 + n_1n_2 \sum_{xy\notin G_1} d_{G_1}(x) \sum_{uv\in G_2} d_{G_2}(v)\\
&- \; n_2 \sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\in G_2} d_{G_2}(v) + n_1n_2 \sum_{xy\notin G_1} d_{G_1}(y) \sum_{uv\in G_2} d_{G_2}(u)\\
&+ \; n_1^2 \sum_{xy\notin G_1} 1 \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v) - n_1 \sum_{xy\notin G_1} d_{G_1}(y) \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)\\
&- \; n_2 \sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\in G_2} d_{G_2}(u) - n_1 \sum_{xy\notin G_1} d_{G_1}(x) \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)\\
&+ \; \sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)\\
&= \; 2n_2^2m_2[2\overline{M}_2(G_1) + M_1(G_1)] + n_1n_2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)]M_1(G_2)\\
&- \; n_2M_1(G_2)[2\overline{M}_2(G_1) + M_1(G_1)] + n_1n_2[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)]M_1(G_2)\\
&+ \; 2n_1^2M_2(G_2)(2\overline{m}_1 + n_1) - 2n_1[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)]M_2(G_2)\\
&- \; n_2M_1(G_2)[2\overline{M}_2(G_1) + M_1(G_1)] - 2n_1[2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)]M_2(G_2)\\
&+ \; 2M_2(G_2)[2\overline{M}_2(G_1) + M_1(G_1)]
\end{aligned}
$$

$$
\begin{aligned}
C_2 &= \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1\vee G_2}((x,u)(y,v))d_{G_1\vee G_2}(x,u)d_{G_1\vee G_2}(y,v)\\
&= \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1\vee G_2}(x,u)d_{G_1\vee G_2}(y,v)\\
&= \sum_{xy\in G_1} \sum_{uv\notin G_2} [n_2d_{G_1}(x) + n_1d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u)][n_2d_{G_1}(y) + n_1d_{G_2}(v) - d_{G_1}(y)d_{G_2}(v)]\\
&= \sum_{xy\in E(G_1)} \sum_{uv\notin G_2} \Big[n_2^2d_{G_1}(x)d_{G_1}(y) + n_1n_2d_{G_1}(x)d_{G_2}(v) - n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)\\
&+ \; n_1n_2d_{G_1}(y)d_{G_2}(u) + n_1^2d_{G_2}(u)d_{G_2}(v) - n_1d_{G_1}(y)d_{G_2}(u)d_{G_2}(v) - n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- \; n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v) + d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= \; n_2^2 \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y) + n_1n_2 \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1}(x)d_{G_2}(v)\\
&- \; n_2 \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(v) + n_1n_2 \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1}(y)d_{G_2}(u)
\end{aligned}
$$

$$\begin{aligned}
&+ \; n_1^2 \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v) - n_1 \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- \; n_2 \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u) - n_1 \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)\\
&+ \; \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= \; n_2^2 \sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2} 1 + n_1n_2 \sum_{xy\in G_1} d_{G_1}(x) \sum_{uv\notin G_2} d_{G_2}(v)\\
&- \; n_2 \sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(v) + n_1n_2 \sum_{xy\in G_1} d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(u)\\
&+ \; n_1^2 \sum_{xy\in G_1} 1 \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v) - n_1 \sum_{xy\in G_1} d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)\\
&- \; n_2 \sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(u) - n_1 \sum_{xy\in G_1} d_{G_1}(x) \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)\\
&+ \; \sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)\\
&= \; 2n_2^2M_2(G_1)[2\overline{m}_2 + n_2] + n_1n_2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)]M_1(G_1)\\
&- \; 2n_2M_2(G_1)[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)] + n_1n_2[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)]M_1(G_1)\\
&+ \; 2n_1^2m_1(2\overline{M}_2(G_2) + M_1(G_2)) - n_1[2\overline{M}_2(G_2) + M_1(G_2)]M_1(G_1)\\
&- \; 2n_2M_2(G_1)[2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G_2})] - n_1[2\overline{M}_2(G_2) + M_1(G_2)]M_1(G_1)\\
&+ \; 2M_2(G_1)[2\overline{M}_2(G_2) + M_1(G_2)]
\end{aligned}$$

$$\begin{aligned}
C_3 &= \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1\vee G_2}\Big((x,u)(y,v)\Big) d_{G_1\vee G_2}(x,u)d_{G_1\vee G_2}(y,v)\\
&= 1 \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1\vee G_2}(x,u)d_{G_1\vee G_2}(y,v)\\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} \Big[n_2d_{G_1}(x) + n_1d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u)\Big]\Big[n_2d_{G_1}(y) + n_1d_{G_2}(v) - d_{G_1}(y)d_{G_2}(v)\Big]\\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} \Big[n_2^2d_{G_1}(x)d_{G_1}(y) + n_1n_2d_{G_1}(x)d_{G_2}(v) - n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)\\
&+ \; n_1n_2d_{G_1}(y)d_{G_2}(u) + n_1^2d_{G_2}(u)d_{G_2}(v) - n_1d_{G_1}(y)d_{G_2}(u)d_{G_2}(v) - n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- \; n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v) + d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= \; n_2^2 \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y) + n_1n_2 \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1}(x)d_{G_2}(v)\\
&- \; n_2 \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(v) + n_1n_2 \sum_{xy\in G_1} \sum_{uv\in G_2} d_{G_1}(y)d_{G_2}(u)
\end{aligned}$$

$$
\begin{aligned}
&+n_1^2\sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)-n_1\sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&-n_2\sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)-n_1\sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)\\
&+\sum_{xy\in G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
=\;& n_2^2\sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2}1+n_1n_2\sum_{xy\in G_1} d_{G_1}(x)\sum_{uv\in G_2} d_{G_2}(v)\\
-\;& n_2\sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(v)+n_1n_2\sum_{xy\in G_1} d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(u)\\
+\;& n_1^2\sum_{xy\in G_1}1\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)-n_1\sum_{xy\in G_1} d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)\\
-\;& n_2\sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(u)-n_1\sum_{xy\in G_1} d_{G_1}(x)\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)\\
+\;& \sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)\\
=\;& 4n_2^2m_2M_2(G_1)+n_1n_2M_1(G_1)M_1(G_2)-2n_2M_2(G_1)M_1(G_2)+n_1n_2M_1(G_1)M_1(G_2)\\
+\;& 4n_1^2m_1M_2(G_2)-2n_1M_1(G_1)M_2(G_2)-2n_2M_2(G_1)M_1(G_2)-2n_1M_1(G_1)M_2(G_2)\\
+\;& 4M_2(G_1)M_2(G_2)
\end{aligned}
$$

$$
\begin{aligned}
C_4 =\;& \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{(G_1\vee G_2)}((x,u)(y,v))d_{(G_1\vee G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
=\;& \sum_{xy\notin G_1}\Big\{\sum_{uv\notin G_2,u\neq v} d_{(G_1\vee G_2)}((x,u)(y,v))d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
+\;& \sum_{uv\notin G_2,u=v} d_{(G_1\vee G_2)}((x,u)(y,v))d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\Big\}\\
=\;& \sum_{xy\notin G_1}\sum_{uv\notin G_2,u\neq v} d_{(G_1\vee G_2)}((x,u)(y,v))d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
+\;& \sum_{xy\notin G_1}\sum_{uv\notin G_2,u=v} d_{(G_1\vee G_2)}((x,u)(y,v))d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
=\;& \sum_{xy\notin G_1,x\neq y}\sum_{uv\notin G_2,u\neq v} d_{(G_1\vee G_2)}((x,u)(y,v))d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
+\;& \sum_{xy\notin G_1,x=y}\sum_{uv\notin G_2,u\neq v} d_{(G_1\vee G_2)}((x,u)(y,v))d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
+\;& \sum_{xy\notin G_1,x\neq y}\sum_{uv\notin G_2,u=v} d_{(G_1\vee G_2)}((x,u)(y,v))d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
+\;& \sum_{xy\notin G_1,x=y}\sum_{uv\notin G_2,u=v} d_{(G_1\vee G_2)}((x,u)(y,v))d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\sum_{xy\notin G_1, x\neq y}\ \sum_{uv\notin G_2, u\neq v} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x=y}\ \sum_{uv\notin G_2, u\neq v} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x\neq y}\ \sum_{uv\notin G_2, u=v} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)+0\\
&= 2\sum_{xy\notin G_1, x\neq y}\ \sum_{uv\notin G_2, u\neq v} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x=y}\ \sum_{uv\notin G_2, u\neq v} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x\neq y}\ \sum_{uv\notin G_2, u=v} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x=y}\ \sum_{uv\notin G_2, u=v} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
&- 2\sum_{xy\notin G_1, x=y}\ \sum_{uv\notin G_2, u=v} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
&= 2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
&- 2\sum_{xy\notin G_1, x=y}\ \sum_{uv\notin G_2, u=v} d_{(G_1\vee G_2)}(x,u)d_{(G_1\vee G_2)}(y,v)\\
&= 2C_5-2C_6, \text{where } C_5 \text{ and } C_6 \text{ are terms of the above sums taken in order.}
\end{aligned}
$$

$$
\begin{aligned}
C_5 &= \sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{G_1\vee G_2}(x,u)d_{G_1\vee G_2}(y,v)\\
&= \sum_{xy\notin G_1}\ \sum_{uv\notin G_2}\Big[n_2d_{G_1}(x)+n_1d_{G_2}(u)-d_{G_1}(x)d_{G_2}(u)\Big]\Big[n_2d_{G_1}(y)+n_1d_{G_2}(v)-d_{G_1}(y)d_{G_2}(v)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{xy\notin G_1}\ \sum_{uv\notin G_2}\Big[n_2^2d_{G_1}(x)d_{G_1}(y)+n_1n_2d_{G_1}(x)d_{G_2}(v)-n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)\\
&+ n_1n_2d_{G_1}(y)d_{G_2}(u)+n_1^2d_{G_2}(u)d_{G_2}(v)-n_1d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)-n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= n_2^2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)+n_1n_2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{G_1}(x)d_{G_2}(v)\\
&- n_2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+n_1n_2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{G_1}(y)d_{G_2}(u)\\
&+ n_1^2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)-n_1\sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- n_2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)-n_1\sum_{xy\notin G_1}\ \sum_{uv\notin G_2} d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= n_2^2\sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2}1 + n_1n_2\sum_{xy\notin G_1} d_{G_1}(x)\sum_{uv\notin G_2} d_{G_2}(v)\\
&- n_2\sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(v) + n_1n_2\sum_{xy\notin G_1} d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(u)\\
&\quad +n_1^2\sum_{xy\notin G_1}1\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v) - n_1\sum_{xy\notin G_1} d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)\\
&- n_2\sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(u) - n_1\sum_{xy\notin G_1} d_{G_1}(x)\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)\\
&+ \sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)\\
&= n_2^2\Big[2\overline{M}_2(G_1)) + M_1(G_1)\Big](2\overline{m}_2 + n_2)\\
&+ n_1n_2\Big[(n_1-1)2\overline{m}_1 - M_1(\overline{G}_1) + 2m_1\Big]\Big((n_2-1)2\overline{m}_2 - M_1(\overline{G}_2) + 2m_2\Big)\\
&- n_2\Big[2\overline{M}_2(G_1) + M_1(G_1)\Big]\Big((n_2-1)2\overline{m}_2 - M_1(\overline{G}_2) + 2m_2\Big)\\
&+ n_1n_2\Big[(n_1-1)2\overline{m}_1 - M_1(\overline{G}_1) + 2m_1\Big]\Big((n_2-1)2\overline{m}_2 - M_1(\overline{G}_2) + 2m_2\Big)\\
&+ n_1^2\Big[2\overline{m}_1 + n_1\Big]\Big(2\overline{M}_2(G_2) + M_1(G_2)\Big) - n_1\Big[2\overline{M}_2(G_2) + M_1(G_2)\Big]\Big((n_1-1)2\overline{m}_1 - M_1(\overline{G}_1) + 2m_1\Big)\\
&- n_2\Big[2\overline{M}_2(G_1) + M_1(G_1)\Big]\Big((n_2-1)2\overline{m}_2 - M_1(\overline{G}_2) + 2m_2\Big)\\
&- n_1\Big[2\overline{M}_2(G_2) + M_1(G_2)\Big]\Big((n_1-1)2\overline{m}_1 - M_1(\overline{G}_1) + 2m_1\Big)\\
&+ \Big[2\overline{M}_2(G_1) + M_1(G_1)\Big]\Big(2\overline{M}_2(G_2) + M_1(G_2)\Big)
\end{aligned}
$$

$$
\begin{aligned}
C_6 &= \sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v} d_{G_1\vee G_2}(x,u)d_{G_1\vee G_2}(y,v)\\
&= \sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v} [n_2d_{G_1}(x) + n_1d_{G_2}(u) - d_{G_1}(x)d_{G_2}(u)][n_2d_{G_1}(y) + n_1d_{G_2}(v)\\
&\quad -d_{G_1}(y)d_{G_2}(v)]\\
&= \sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v}\Big[n_2^2d_{G_1}(x)d_{G_1}(y) + n_1n_2d_{G_1}(x)d_{G_2}(v) - n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)\\
&+ n_1n_2d_{G_1}(y)d_{G_2}(u) + n_1^2d_{G_2}(u)d_{G_2}(v) - n_1d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u) - n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v) + d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= n_2^2\sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_1}(y) + n_1n_2\sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_2}(v)
\end{aligned}
$$

$$\begin{aligned}
&- n_2 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_1}(y)d_{G_2}(v) + n_1n_2 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} d_{G_1}(y)d_{G_2}(u)\\
&+ n_1^2 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v) - n_1 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- n_2 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u) - n_1 \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)\\
&+ \sum_{xy\notin G_1, x=y} \sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= n_2^2 \sum_{xy\notin G_1, x=y} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2, u=v} 1 + n_1n_2 \sum_{xy\notin G_1, x=y} d_{G_1}(x) \sum_{uv\notin G_2, u=v} d_{G_2}(v)\\
&- n_2 \sum_{xy\notin G_1, x=y} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(v) + n_1n_2 \sum_{xy\notin G_1, x=y)} d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(u)\\
&+ n_1^2 \sum_{xy\notin G_1, x=y} 1 \sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v) - n_1 \sum_{xy\notin G_1 x=y} d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v)\\
&- n_2 \sum_{xy\notin G_1, x=y} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(u) - n_1 \sum_{xy\notin G_1, x=y} d_{G_1}(x) \sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v)\\
&+ \sum_{xy\notin G_1, x=y)} d_{G_1}(x)d_{G_1}(y) \sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v)]\\
&= n_2^3M_1(G_1) + 4n_1n_2m_1m_2 - 2n_2m_2M_1(G_1) + 4n_1n_2m_1m_2\\
&+ n_1^3M_1(G_2) - 2n_1m_1M_1(G_2) - 2n_2m_2M_1(G_1) - 2n_1m_1M_1(G_2) + M_1(G_1)M_1(G_2)
\end{aligned}$$

$$\begin{aligned}
\therefore\ 2 \times Gut(G_1 \vee G_2) &= C_1 + C_2 + C_3 + C_4\\
&= C_1 + C_2 + C_3 + 2C_5 - 2C_6
\end{aligned}$$

By substituting C_1, C_2, C_3, C_5 and C_6, the desired result follows after simple calculation.

∎

6 Symmetric difference

In the following theorem, we obtain the Gutman index of the symmetric difference of two graphs.

Theorem 6.1.

$$2 \times Gut(G_1 \oplus G_2) = \Big(2\overline{M}_2(G_1) + M_1(G_1)\Big)\Big[2n_2^2m_2 - 4n_2M_1(G_2) + 8M_2(G_2) + 2n_2^2(2\overline{m}_2 + n_2)\Big]$$

$$
\begin{aligned}
&+ \Big(2\overline{M}_2(G_2) + M_1(G_2)\Big)\Big[2n_1^2 m_1 - 4n_1 M_1(G_1) + 8M_2(G_1) + 2n_1^2(2\overline{m}_1 + n_1)\Big] \\
&+ \Big(2\overline{m}_1(n_1-1) + 2m_1 - M_1(\overline{G}_1)\Big)\Big[2n_1n_2M_1(G_2) - 8n_1M_2(G_2) \\
&- 8n_1\Big\{2\overline{M}_2(G_2) + M_1(G_2)\Big\}\Big] + \Big(2\overline{m}_2(n_2-1) + 2m_2 - M_1(\overline{G}_2)\Big)\Big[2n_1n_2M_1(G_1) \\
&- 8n_2M_2(G_1) - 8n_2\Big\{2\overline{M}_2(G_1) + M_1(G_1)\Big\}\Big] + 2n_1^2M_2(G_2)(2\overline{m}_1 + n_1) \\
&+ 2n_2^2M_2(G_1)[2\overline{m}_2 + n_2] + 8n_2^2m_2M_2(G_1) + 4n_1n_2M_1(G_1)M_1(G_2) + 8n_1^2m_1M_2(G_2) \\
&- 16n_2M_2(G_1)M_1(G_2) - 16n_1M_1(G_1)M_2(G_2) + 32M_2(G_1)M_2(G_2) \\
&+ 4n_1n_2\Big[(n_1-1)2\overline{m}_1 - M_1(\overline{G}_1) + 2m_1\Big]\Big[(n_2-1)2\overline{m}_2 - M_1(\overline{G}_2) + 2m_2\Big] \\
&- 2n_1^3M_1(G_2) + 16n_1m_1M_1(G_2) - 8M_1(G_1)M_1(G_2) + 16n_2m_2M_1(G_1) \\
&+ 8\Big[2\overline{M}_2(G_1) + M_1(G_1)\Big]\Big[2\overline{M}_2(G_2) + M_1(G_2)\Big] - 2n_2^3M_1(G_1) - 16n_1n_2m_1m_2
\end{aligned}
$$

Proof:

$$
2 \times Gut(G_1 \oplus G_2) = \sum_{x,y\in V(G_1)} \sum_{u,v\in V(G_2)} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)
$$

$$
\begin{aligned}
&= \sum_{x,y\in V(G_1)} \Big\{ \sum_{uv\in G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v) \\
&+ \sum_{uv\notin G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\Big\} \\
&= \sum_{x,y\in V(G_1)} \sum_{uv\in G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v) \\
&+ \sum_{x,y\in V(G_1)} \sum_{uv\notin G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v) \\
&= \sum_{xy\in G_1} \sum_{uv\in G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v) \\
&+ \sum_{xy\notin G_1} \sum_{uv\in G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v) \\
&+ \sum_{xy\in G_1} \sum_{uv\notin G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v) \\
&+ \sum_{xy\notin G_1} \sum_{uv\notin G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)
\end{aligned}
$$

$= B_3 + B_1 + B_2 + B_4$, where B_3, B_1, B_2 and B_4 are terms of the above sums taken in order.

$$
\begin{aligned}
B_1 &= \sum_{xy\notin G_1}\sum_{uv\in G_2} d_{(G_1\oplus G_2)}\Big((x,u)(y,v)\Big) d_{(G_1\oplus G_2)}(x,u) d_{(G_1\oplus G_2)}(y,v)\\
&= 1\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{(G_1\oplus G_2)}(x,u) d_{(G_1\oplus G_2)}(y,v)\\
&= \sum_{xy\notin G_1}\sum_{uv\in G_2}\Big[n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\Big]\Big[n_2d_{G_1}(y)+n_1d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\Big]\\
&= \sum_{xy\notin G_1}\sum_{uv\in G_2}\Big[n_2^2d_{G_1}(x)d_{G_1}(y)+n_1n_2d_{G_1}(x)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)\\
&+ n_1n_2d_{G_1}(y)d_{G_2}(u)+n_1^2d_{G_2}(u)d_{G_2}(v)-2n_1d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- 2n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+4d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= n_2^2\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y)+n_1n_2\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+n_1n_2\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(y)d_{G_2}(u)\\
&+ n_1^2\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)-2n_1\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\notin G_1}\sum_{uv\in G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)]\\
&= n_2^2\sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2}1+n_1n_2\sum_{xy\notin G_1} d_{G_1}(x)\sum_{uv\in G_2} d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(v)+n_1n_2\sum_{xy\notin G_1} d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(u)\\
&+ n_1^2\sum_{xy\notin G_1}1\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\notin G_1} d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(u)-2n_1\sum_{xy\notin G_1} d_{G_1}(x)\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\notin G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2} d_{G_2}(u)d_{G_2}(v)]\\
&= 2n_2^2m_2[2\overline{M}_2(G_1)+M_1(G_1)]+n_1n_2[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)]M_1(G_2)\\
&- 2n_2M_1(G_2)[2\overline{M}_2(G_1)+M_1(G_1)]+n_1n_2[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)]M_1(G_2)\\
&+ 2n_1^2M_2(G_2)(2\overline{m}_1+n_1)-2n_1[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)]M_2(G_2)\\
&- 2n_2M_1(G_2)[2\overline{M}_2(G_1)+M_1(G_1)]-2n_1[2\overline{m}_1(n_1-1)+2m_1-M_1(\overline{G}_1)]M_2(G_2)\\
&+ 8M_2(G_2)[2\overline{M}_2(G_1)+M_1(G_1)]
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= \sum_{xy\in G_1}\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}[(x,u)(y,v)]d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2} [n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)][n_2d_{G_1}(y)+n_1d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)]\\
&= \sum_{xy\in G_1}\sum_{uv\notin G_2} \Big[n_2^2d_{G_1}(x)d_{G_1}(y)+n_1n_2d_{G_1}(x)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)\\
&+ n_1n_2d_{G_1}(y)d_{G_2}(u)+n_1^2d_{G_2}(u)d_{G_2}(v)-2n_1d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- 2n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+4d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= n_2^2\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)+n_1n_2\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(x)d_{G_2}(v)\\
&- 2n_2\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+n_1n_2\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(y)d_{G_2}(u)\\
&+ 2n_1^2\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)-2n_1\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\in G_1}\sum_{uv\notin G_2} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= n_2^2\sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2} 1+n_1n_2\sum_{xy\in G_1} d_{G_1}(x)\sum_{uv\notin G_2} d_{G_2}(v)\\
&- 2n_2\sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(v)+n_1n_2\sum_{xy\in G_1} d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(u)\\
&+ n_1^2\sum_{xy\in G_1} 1\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\in G_1} d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(u)-2n_1\sum_{xy\in G_1} d_{G_1}(x)\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\in G_1} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2} d_{G_2}(u)d_{G_2}(v)\\
&= 2n_2^2M_2(G_1)[2\overline{m}_2+n_2]+n_1n_2[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)]M_1(G_1)\\
&- 4n_2M_2(G_1)[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)]+n_1n_2[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G}_2)]M_1(G_1)\\
&+ 2n_1^2m_1(2\overline{M}_2(G_2)+M_1(G_2))-2n_1[2\overline{M}_2(G_2)+M_1(G_2)]M_1(G_1)\\
&- 4n_2M_2(G_1)[2\overline{m}_2(n_2-1)+2m_2-M_1(\overline{G_2})]-2n_1[2\overline{M}_2(G_2)+M_1(G_2)]M_1(G_1)\\
&+ 8M_2(G_1)[2\overline{M}_2(G_2)+M_1(G_2)]
\end{aligned}
$$

$$
\begin{aligned}
B_3 &= \sum_{xy\in G_1}\sum_{uv\in G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&= \sum_{xy\in G_1}\sum_{uv\in G_2} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&= 2\sum_{xy\in G_1}\sum_{uv\in G_2}\Big[n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\Big]\Big[n_2d_{G_1}(y)+n_1d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\Big]\\
&= 2\sum_{xy\in G_1}\sum_{uv\in G_2}\Big[n_2^2d_{G_1}(x)d_{G_1}(y)+n_1n_2d_{G_1}(x)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)\\
&+ n_1n_2d_{G_1}(y)d_{G_2}(u)+n_1^2d_{G_2}(u)d_{G_2}(v)-2n_1d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- 2n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+4d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= 2\Big[n_2^2\sum_{xy\in G_1}\sum_{uv\in G_2}d_{G_1}(x)d_{G_1}(y)+n_1n_2\sum_{xy\in G_1}\sum_{uv\in G_2}d_{G_1}(x)d_{G_2}(v)\\
&- 2n_2\sum_{xy\in G_1}\sum_{uv\in G_2)}d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+n_1n_2\sum_{xy\in G_1}\sum_{uv\in G_2}d_{G_1}(y)d_{G_2}(u)\\
&+ n_1^2\sum_{xy\in G_1}\sum_{uv\in G_2}d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\in G_1}\sum_{uv\in G_2}d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\in G_1}\sum_{uv\in G_2}d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)-2n_1\sum_{xy\in G_1}\sum_{uv\in G_2}d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\in G_1}\sum_{uv\in G_2}d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= 2\Big[n_2^2\sum_{xy\in G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2}1+n_1n_2\sum_{xy\in G_1}d_{G_1}(x)\sum_{uv\in G_2}d_{G_2}(v)\\
&- 2n_2\sum_{xy\in G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2}d_{G_2}(v)+n_1n_2\sum_{xy\in G_1}d_{G_1}(y)\sum_{uv\in G_2}d_{G_2}(u)\\
&+ n_1^2\sum_{xy\in G_1}1\sum_{uv\in G_2}d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\in G_1}d_{G_1}(y)\sum_{uv\in G_2}d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\in G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2}d_{G_2}(u)-2n_1\sum_{xy\in G_1}d_{G_1}(x)\sum_{uv\in G_2}d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\in G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\in G_2}d_{G_2}(u)d_{G_2}(v)\Big]\\
&= 2\Big[4n_2^2m_2M_2(G_1)+n_1n_2M_1(G_1)M_1(G_2)-4n_2M_2(G_1)M_1(G_2)+n_1n_2M_1(G_1)M_1(G_2)\\
&+ 4n_1^2m_1M_2(G_2)-4n_1M_1(G_1)M_2(G_2)-4n_2M_2(G_1)M_1(G_2)-4n_1M_1(G_1)M_2(G_2)\\
&+ 16M_2(G_1)M_2(G_2)\Big]
\end{aligned}
$$

$$
B_4 = \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)
$$

$$
\begin{aligned}
&= \sum_{xy\notin G_1}\Big\{\sum_{uv\notin G_2, u\neq v} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ \sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\Big\}\\
&= \sum_{xy\notin G_1}\sum_{uv\notin G_2, u\neq v} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ \sum_{xy\notin G_1}\sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&= \sum_{xy\notin G_1, x\neq y}\sum_{uv\notin G_2, u\neq v} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ \sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u\neq v} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ \sum_{xy\notin G_1, x\neq y}\sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ \sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}((x,u)(y,v))d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&= 2\sum_{xy\notin G_1, x\neq y}\sum_{uv\notin G_2, u\neq v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u\neq v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x\neq y}\sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v) + 0\\
&= 2\sum_{xy\notin G_1, x\neq y}\sum_{uv\notin G_2, u\neq v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u\neq v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x\neq y}\sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&+ 2\sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&- 2\sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&= 2\sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&- 2\sum_{xy\notin G_1, x=y}\sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&= 2B_5 - 2B_6, \text{where } B_5 \text{ and } B_6 \text{ are terms of the above sums taken in order.}
\end{aligned}
$$

$$
B_5 = \sum_{xy\notin G_1}\sum_{uv\notin G_2} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)
$$

$$
\begin{aligned}
&= \sum_{xy\notin G_1}\ \sum_{uv\notin G_2}\Big[n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\Big]\Big[n_2d_{G_1}(y)+n_1d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\Big]\\
&= \sum_{xy\notin G_1}\ \sum_{uv\notin G_2}\Big[n_2^2d_{G_1}(x)d_{G_1}(y)+n_1n_2d_{G_1}(x)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)\\
&+ n_1n_2d_{G_1}(y)d_{G_2}(u)+n_1^2d_{G_2}(u)d_{G_2}(v)-2n_1d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- 2n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+4d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= n_2^2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2}d_{G_1}(x)d_{G_1}(y)+n_1n_2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2}d_{G_1}(x)d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2}d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+n_1n_2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2}d_{G_1}(y)d_{G_2}(u)\\
&+ n_1^2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2}d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\notin G_1}\ \sum_{uv\notin G_2}d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1}\ \sum_{uv\notin G_2}d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)-2n_1\sum_{xy\notin G_1}\ \sum_{uv\notin G_2}d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\notin G_1}\ \sum_{uv\notin G_2}d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= n_2^2\sum_{xy\notin G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2}1+n_1n_2\sum_{xy\notin G_1}d_{G_1}(x)\sum_{uv\notin G_2}d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(v)+n_1n_2\sum_{xy\notin G_1}d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(u)\\
&+ n_1^2\sum_{xy\notin G_1}1\sum_{uv\notin G_2}d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\notin G_1}d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(u)-2n_1\sum_{xy\notin G_1}d_{G_1}(x)\sum_{uv\notin G_2}d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\notin G_1}d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2}d_{G_2}(u)d_{G_2}(v)\\
&= n_2^2\Big[2\overline{M}_2(G_1))+M_1(G_1)\Big](2\overline{m}_2+n_2)\\
&+ n_1n_2\Big[(n_1-1)2\overline{m}_1-M_1(\overline{G_1})+2m_1\Big]\Big((n_2-1)2\overline{m}_2-M_1(\overline{G_2})+2m_2\Big)\\
&- 2n_2\Big[2\overline{M}_2(G_1)+M_1(G_1)\Big]\Big((n_2-1)2\overline{m}_2-M_1(\overline{G_2})+2m_2\Big)\\
&+ n_1n_2\Big[(n_1-1)2\overline{m}_1-M_1(\overline{G_1})+2m_1\Big]\Big((n_2-1)2\overline{m}_2-M_1(\overline{G_2})+2m_2\Big)\\
&+ n_1^2[2\overline{m}_1+n_1][2\overline{M}_2(G_2)+M_1(G_2)]\\
&- 2n_1\Big[2\overline{M}_2(G_2)+M_1(G_2)\Big]\Big((n_1-1)2\overline{m}_1-M_1(\overline{G_1})+2m_1\Big)\\
&- 2n_2\Big[2\overline{M}_2(G_1)+M_1(G_1)\Big]\Big((n_2-1)2\overline{m}_2-M_1(\overline{G_2})+2m_2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \; 2n_1\Big[2\overline{M}_2(G_2)+M_1(G_2)\Big]\Big((n_1-1)2\overline{m}_1 - M_1(\overline{G_1}) + 2m_1\Big)\\
&+ \; 4[2\overline{M}_2(G_1)+M_1(G_1)][2\overline{M}_2(G_2)+M_1(G_2)]
\end{aligned}
$$

$$
\begin{aligned}
B_6 &= \sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{(G_1\oplus G_2)}(x,u)d_{(G_1\oplus G_2)}(y,v)\\
&= \sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v}\Big[n_2d_{G_1}(x)+n_1d_{G_2}(u)-2d_{G_1}(x)d_{G_2}(u)\Big]\\
&\quad \Big[n_2d_{G_1}(y)+n_1d_{G_2}(v)-2d_{G_1}(y)d_{G_2}(v)\Big]\\
&= \sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v}\Big[n_2^2d_{G_1}(x)d_{G_1}(y)+n_1n_2d_{G_1}(x)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)\\
&+ n_1n_2d_{G_1}(y)d_{G_2}(u)+n_1^2d_{G_2}(u)d_{G_2}(v)-2n_1d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)-2n_2d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)\\
&- 2n_1d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)+4d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\Big]\\
&= n_2^2\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_1}(y)+n_1n_2\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_1}(y)d_{G_2}(v)+n_1n_2\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{G_1}(y)d_{G_2}(u)\\
&+ n_1^2\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)-2n_1\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\notin G_1, x=y}\;\sum_{uv\notin G_2, u=v} d_{G_1}(x)d_{G_1}(y)d_{G_2}(u)d_{G_2}(v)\\
&= n_2^2\sum_{xy\notin G_1, x=y} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2, u=v} 1+n_1n_2\sum_{xy\notin G_1, x=y} d_{G_1}(x)\sum_{uv\notin G_2, u=v} d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1, x=y} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2, u=v} d_{G_2}(v)+n_1n_2\sum_{xy\notin G_1, x=y} d_{G_1}(y)\sum_{uv\notin G_2, u=v} d_{G_2}(u)\\
&+ n_1^2\sum_{xy\notin G_1, x=y} 1\sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v)-2n_1\sum_{xy\notin G_1, x=y} d_{G_1}(y)\sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v)\\
&- 2n_2\sum_{xy\notin G_1, x=y} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2, u=v} d_{G_2}(u)-2n_1\sum_{xy\notin G_1, x=y} d_{G_1}(x)\sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v)\\
&+ 4\sum_{xy\notin G_1, x=y} d_{G_1}(x)d_{G_1}(y)\sum_{uv\notin G_2, u=v} d_{G_2}(u)d_{G_2}(v)
\end{aligned}
$$

$$
\begin{aligned}
&= n_2^3M_1(G_1)+4n_1n_2m_1m_2-4n_2m_2M_1(G_1)+4n_1n_2m_1m_2\\
&+ n_1^3M_1(G_2)-4n_1m_1M_1(G_2)-4n_2m_2M_1(G_1)-4n_1m_1M_1(G_2)+4M_1(G_1)M_1(G_2)
\end{aligned}
$$

$$\therefore\ 2 \times Gut(G_1 \oplus G_2) = B_1 + B_2 + B_3 + B_4$$
$$= B_1 + B_2 + B_3 + 2B_5 - 2B_6$$

By substituting B_1, B_2, B_3, B_5 and B_6, the desired result follows after simple calculation.

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