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On Product of Randić and Sum-Connectivity Energy of Graphs

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Abstract

In this paper we define the product of Randić and sum-connectivity energy of a graph. Then we obtain upper and lower bounds for $E_{prs}(G)$, product of Randić and sum-connectivity energy of a graph. Further we compute the product of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the $(S_m \wedge P_2)$ graph.

Key words: Product of Randić and sum-connectivity energy

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1 Introduction

In 2010, Bo Zhou and Nenad Trinajstic [3] have introduced the sum-connectivity energy of a graph as follows. Let G be a simple graph and let $v_1, v_2, \ldots, v_n$ be its vertices. For $i = 1, 2, \ldots, n$, let d_i denote the degree of the vertex v_i. Then the sum-connectivity matrix of G is defined as $R = (R_{ij})$, where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i + d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The sum-connectivity energy of G is defined as the sum of absolute values of the eigenvalues of the sum-connectivity matrix of G arranged in a non-increasing order.

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In the same year, Burcu Bozkurt, Dilek Güngör, Gutman and Sinan Çevik [2], have defined the Randić energy of a graph G as the sum of the absolute values of the eigenvalues of the Randić matrix (R_{ij}) where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

Motivated by these works, we introduce the product of Randić and sum-connectivity energy of a simple graph G as follows. The product of Randić and sum-connectivity adjacency matrix of G is the $n \times n$ matrix $A_{prs} = (a_{ij})$ where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i^2 d_j + d_i d_j^2}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

The eigenvalues of the graph G are the eigenvalues of A_{prs}. Since A_{prs} is real and symmetric, its eigenvalues are real numbers which are denoted by $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n$. Then the product of Randić and sum-connectivity energy of G is defined as

$$E_{prs}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

Since A_{psr} is a real symmetric matrix, we have

$$\sum_{i=1}^{n} \lambda_i = tr(A_{psr}) = 0 \tag{1}$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = tr(A_{psr}^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 = 2 \sum_{i \sim j} \frac{1}{d_i^2 d_j + d_i d_j^2} \tag{2}$$

In this paper we obtain the upper and lower bounds for $E_{psr}(G)$ and compute the $E_{prs}(G)$ of complete graph, star graph, complete bipartite graph, the $(S_m \wedge P_2)$ graph.

2 Upper and lower bounds for $E_{prs}(G)$

In this section we obtain Upper and lower bounds for $E_{psr}(G)$.

Theorem 2.1. Let G be a simple graph of order n with no isolated vertices. Then

$$E_{prs}(G) \leq \sqrt{2n \sum_{i \sim j} \frac{1}{d_i^2 d_j + d_i d_j^2}}. \tag{3}$$

Proof: Let $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$, be the eigenvalues of A_{prs}. Then using (2) and the Cauchy-Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right)$$

with $a_i = 1$, $b_i =\mid \lambda_i \mid$, we obtain

$$E_{prs}(G) = \sum_{i=1}^{n} |\lambda_i| = \sqrt{(\sum_{i=1}^{n} |\lambda_i|)^2} \leq \sqrt{n \sum_{i=1}^{n} \lambda_i^2} = \sqrt{2n \sum_{i \sim j} \frac{1}{d_i^2 d_j + d_i d_j^2}}.$$

■

Theorem 2.2. Let G be a simple graph of order n. Then

$$E_{prs}(G) \geq 2\sqrt{\sum_{i \sim j} \frac{1}{d_i^2 d_j + d_i d_j^2}}. \tag{4}$$

Proof: From (1), we have

$$\sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j = 0$$

and therefore

$$-\sum_{i=1}^{n} \lambda_i^2 = 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j. \tag{5}$$

Thus

$$(E_{prs}(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2 = \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1\le i<j\le n} |\lambda_i\lambda_j|$$

$$\ge \sum_{i=1}^{n} \lambda_i^2 + 2 \left| \sum_{1\le i<j\le n} \lambda_i\lambda_j \right|$$

$$= 2\sum_{i=1}^{n} \lambda_i^2, \ \ on \ \ using\, (5).$$

This together with (2) implies that

$$(E_{prs}(G))^2 \ge 4 \sum_{i\sim j} \frac{1}{d_i^2 d_j + d_i d_j^2},$$

which gives (4). ∎

3 Product of Randić and sum-connectivity energies of some families of graphs

We begin with some basic definitions and notations.

Definition 3.1. [4] A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.

Definition 3.2. [4] A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.

Definition 3.3. [5] The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m) \ and \ v_j v_l \in E(P_2) \ and \ 1 \le i, k \le m + 1, 1 \le j, l \le 2\}$.

Now we compute product of Randić and sum-connectivity energies of complete graph, star graph, complete bipartite graph, the $(S_m \wedge P_2)$ graph.

Theorem 3.4. The product of Randić and sum-connectivity energy of the complete bipartite graph $K_{m,n}$ is $2\sqrt{\frac{mn}{mn(m+n)}}$.

Proof: Let the vertex set of the complete bipartite graph be $V(K_{m,n}) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$. Then the product of Randić and sum-connectivity matrix of complete bipartite graph is given by

$$A_{prs} = \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{\sqrt{mn(m+n)}} & \cdots & \frac{1}{\sqrt{mn(m+n)}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\sqrt{mn(m+n)}} & \cdots & \frac{1}{\sqrt{mn(m+n)}} \\ \frac{1}{\sqrt{mn(m+n)}} & \cdots & \frac{1}{\sqrt{mn(m+n)}} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{mn(m+n)}} & \cdots & \frac{1}{\sqrt{mn(m+n)}} & 0 & \cdots & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{prs}| = \begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{mn(m+n)}} J^T \\ -\frac{1}{\sqrt{mn(m+n)}} J & \lambda I_n \end{vmatrix},$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\frac{1}{\sqrt{mn(m+n)}} J^T \\ -\frac{1}{\sqrt{mn(m+n)}} J & \lambda I_n \end{vmatrix} = 0,$$

which can be written as

$$|\lambda I_m| \left| \lambda I_n - \left(-\frac{1}{\sqrt{mn(m+n)}} J \right) \frac{I_m}{\lambda} \left(-\frac{1}{\sqrt{mn(m+n)}} J^T \right) \right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{(\frac{1}{mn(m+n)})^n} \left| mn(m+n)\lambda^2 I_n - JJ^T \right| = 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{(\frac{1}{mn(m+n)})^n} P_{JJ^T}(mn(m+n))\lambda^2) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_mJ_n$. Thus, we have

$$\frac{\lambda^{m-n}}{(\frac{1}{mn(m+n)})^n} (mn(m+n)\lambda^2 - mn)(mn(m+n)\lambda^2)^{n-1} = 0,$$

which is same as

$$\lambda^{m+n-2}(\lambda^2 - \frac{mn}{mn(m+n)}) = 0.$$

Therefore, the spectrum of $K_{m,n}$ is given by

$$Spec\,(K_{m,n}) = \begin{pmatrix} 0 & \sqrt{\frac{(mn)}{mn(m+n)}} & -\sqrt{\frac{mn}{mn(m+n)}} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Hence the product of Randić and sum-connectivity energy of the complete bipartite graph is

$$E_{prs}(K_{m,n}) = 2\sqrt{\frac{mn}{mn(m+n)}},$$

as desired. ∎

Theorem 3.5. The product of Randić and sum-connectivity energy of the S_n is $\frac{2}{\sqrt{n}}$.

Proof: Let the vertex set of star graph be given by $V(S_n) = \{v_1, v_2, \ldots, v_n\}$. Then product of Randić and sum-connectivity matrix of the star graph S_n is given by

$$A_{prs} = \begin{pmatrix} 0 & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} & \cdots & \frac{1}{\sqrt{(n-1)n}} & \frac{1}{\sqrt{(n-1)n}} \\ \frac{1}{\sqrt{(n-1)n}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{(n-1)n}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{(n-1)n}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{(n-1)n}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{prs}| = \begin{vmatrix} \lambda & -\frac{1}{\sqrt{(n-1)n}} & -\frac{1}{\sqrt{(n-1)n}} & \cdots & -\frac{1}{\sqrt{(n-1)n}} \\ -\frac{1}{\sqrt{(n-1)n}} & \lambda & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{(n-1)n}} & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{(n-1)n}} & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{(n-1)n}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{n(n-1)}$. Then $|\lambda I - A_{prs}| = \phi_n(\mu)\left(\frac{1}{\sqrt{(n-1)n}}\right)^n$,

where $\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ 1 & \mu & 0 & \cdots & 0 & 0 \\ 1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \mu & 0 \\ 1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}$.

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\mu) = (\mu\phi_{n-1}(\mu) - \mu^{n-2}).$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1)).$$

Therefore

$$|\lambda I - A_{prs}| = \left(\frac{1}{\sqrt{(n-1)n}}\right)^n \left[\left((n(n-1))\lambda^2 - (n-1)\right)(\lambda\sqrt{n(n-1)})^{n-2}\right].$$

Thus the characteristic equation is given by

$$\lambda^{n-2}\left(\lambda^2 - \frac{(n-1)}{(n-1)n}\right) = 0.$$

Hence

$$Spec\,(S_n) = \begin{pmatrix} 0 & \frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence the product of Randić and sum-connectivity energy of S_n is

$$E_{prs}(S_n) = \frac{2}{\sqrt{n}}.$$

■

Theorem 3.6. The product of Randić and sum-connectivity energy of K_n is $2\sqrt{\frac{1}{2(n-1)}}$.

Proof: Let the vertex set of Complete graph be given by $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Then the product of Randić and sum-connectivity matrix of the complete graph K_n is given by

$$A_{prs} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2(n-1)^3}} & \cdots & \frac{1}{\sqrt{2(n-1)^3}} \\ \frac{1}{\sqrt{2(n-1)^3}} & 0 & \cdots & \frac{1}{\sqrt{2(n-1)^3}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2(n-1)^3}} & \frac{1}{\sqrt{2(n-1)^3}} & \cdots & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{prs}| = \begin{vmatrix} \lambda & -\frac{1}{\sqrt{2(n-1)^3}} & \cdots & -\frac{1}{\sqrt{2(n-1)^3}} \\ -\frac{1}{\sqrt{2(n-1)^3}} & \lambda & \cdots & -\frac{1}{\sqrt{2(n-1)^3}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{2(n-1)^3}} & -\frac{1}{\sqrt{2(n-1)^3}} & \cdots & \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2(n-1)^3}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{2(n-1)^3}$. Then $|\lambda I - A_{prs}| = \phi_n(\mu)\left(\frac{1}{\sqrt{2(n-1)^3}}\right)^n$,

where $\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}$

$$= \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ 0 & 0 & 0 & \cdots & -1-\mu & \mu+1 \end{vmatrix}$$

$$= (\mu+1) \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} + (\mu+1) \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}.$$

$$\begin{aligned} \phi_n(\mu) &= -(\mu+1)^{n-1} + (\mu+1)\left[(\mu+1)^{n-2}(\mu-(n-2))\right] \\ &= -(\mu+1)^{n-1} + (\mu+1)^{n-1}(\mu-(n-2)). \end{aligned}$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1)),$$

thus the characteristic equation is given by

$$\left(\frac{1}{\sqrt{2(n-1)^3}}\right)^n (\mu+1)^{n-1}(\mu-(n-1)) = 0.$$

Hence

$$Spec\,(K_n) = \begin{pmatrix} \frac{-1}{\sqrt{2(n-1)^3}} & \frac{n-1}{\sqrt{2(n-1)^3}} \\ n-1 & 1 \end{pmatrix}.$$

Hence the product of Randić and sum-connectivity energy of K_n is

$$E_g(K_n) = 2\sqrt{\frac{1}{2(n-1)}}.$$

∎

Theorem 3.7. The product of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ is $4\sqrt{\frac{1}{n}}$.

Proof: Let the vertex set of $(S_m \wedge P_2)$ graph be given by $V(S_m \wedge P_2) = \{v_1, v_2, \ldots, v_{2m+2}\}$.

Then the product of Randić and sum-connectivity matrix of $(S_m \wedge P_2)$ graph is given by

$$A_{prs} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{\sqrt{n(n-1)}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n(n-1)}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n(n-1)}} & \cdots & 0 \\ 0 & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{1}{\sqrt{n(n-1)}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{n(n-1)}} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n(n-1)}} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where $m+1=n$. Its characteristic polynomial is given by

$$|\lambda I - A_{prs}| = \begin{vmatrix} \lambda & \cdots & 0 & 0 & -\frac{1}{\sqrt{n(n-1)}} & -\frac{1}{\sqrt{n(n-1)}} \\ 0 & \cdots & 0 & -\frac{1}{\sqrt{n(n-1)}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda & -\frac{1}{\sqrt{n(n-1)}} & \cdots & 0 \\ 0 & \cdots & -\frac{1}{\sqrt{n(n-1)}} & \lambda & \cdots & 0 \\ -\frac{1}{\sqrt{n(n-1)}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{n(n-1)}} & \cdots & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence the characteristic equation is given by

$$\left(\frac{1}{\sqrt{n(n-1)}}\right)^{2n} \begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

$$where\ \Lambda = \sqrt{n(n-1)}\lambda.$$

Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}.$$

$$= (-1)^{2n+2n}\Lambda \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+2} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+2}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain, after some simplifications

$$\Psi_{2n-1}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda),$$

where $\Theta_n(\Lambda)=\begin{vmatrix} \Lambda & 0 & 0 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}$.

Then

$$\phi_{2n}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned} \phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+2}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= -\Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda). \end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda) = -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda),$$

where $\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}$.

$$\begin{aligned} \phi_{2n}(\Lambda) &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda) \\ &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda) \\ &= (\Lambda^n - (n-1)\Lambda^{n-2})\Theta_n(\Lambda). \end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda) = \Lambda^n - (n-1)\Lambda^{n-2}.$$

Therefore

$$\phi_{2n}(\Lambda) = (\Lambda^n - (n-1)\Lambda^{n-2})^2.$$

Hence characteristic equation becomes

$$\left(\frac{1}{\sqrt{n(n-1)}}\right)^{2n} \phi_{2n}(\Lambda) = 0,$$

which is same as

$$\left(\frac{1}{\sqrt{n(n-1)}}\right)^{2n} (\Lambda^n - (n-1)\Lambda^{n-2})^2 = 0.$$

This reduces to

$$\lambda^{2n-4}((n(n-1))\lambda^2 - (n-1))^2 = 0.$$

Therefore

$$Spec\left((S_m \wedge P_2)\right) = \begin{pmatrix} 0 & \sqrt{\frac{(n-1)}{n(n-1)}} & -\sqrt{\frac{(n-1)}{n(n-1)}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the product of Randić and sum-connectivity energy of $(S_m \wedge P_2)$ graph is

$$E_{prs}((S_m \wedge P_2)) = 4\sqrt{\frac{1}{n}}.$$

■

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