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On Randić Sum Eccentricity Energy of Graphs

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Abstract

In this paper we introduce the concept of Randić sum eccentricity energy of graphs. We obtain upper and lower bounds for Randić sum eccentricity energy of graphs and compute the Randić sum eccentricity of some graphs such as complete graph, star graph, complete bipartite graph, $(S_m \wedge P_2)$ graph and crown graph.

Key words: Randić sum eccentricity energy

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1 Introduction

In 2018, B. Sharada and Mohammad issa ahmed sowaity [1] have introduced the sum eccentricity energy of a simple graph G. The sum eccentricity adjacency matrix of G is the $n \times n$ matrix (a_{ij}), where

$$a_{ij} = \begin{cases} e(v_i) + e(v_j), & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

where $e(v_i)$ is the eccentricity of the vertex v_i. The sum eccentricity energy of G is defined as the sum of absolute values of the eigenvalues of the sum eccentricity adjacency matrix of G.

In 2010, Burcu et al [5], have defined the Randić energy of a graph G as the sum of the absolute values of the eigenvalues of the Randić matrix (R_{ij}) where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

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where d_i denote the degree of the vertex v_i.
Motivated by these works, we have introduced the Randić sum eccentricity energy of a simple graph G as follows. The Randić sum eccentricity energy adjacency matrix of G is the $n \times n$ matrix $A_{rse} = (a_{ij})$, where

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{e(v_i)+e(v_j)}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

where $e(v_i)$ is the eccentricity of the vertex v_i. The Randić sum eccentricity energy of G is defined as the sum of absolute values of the eigenvalues of the Randić sum eccentricity energy adjacency matrix of G.

For example, let G be the cycle on 4 vertices. Then

$$A_{rse} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

Then the characteristic equation is given by $\lambda^4 - \lambda^2$. The eigenvalues are 1, 0, 0, -1 and Randić sum eccentricity energy of G is 2.
Since A_{rse} is real and symmetric, its eigenvalues are real numbers which are denoted by $\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq ... \geq \lambda_n$. Then the Randić sum eccentricity energy of G is defined as

$$E_{rse}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

Since A_{rse} is a real symmetric matrix, we have

$$\sum_{i=1}^{n} \lambda_i = tr(A_{rse}) = 0 \tag{1}$$

and

$$\sum_{i=1}^{n} \lambda_i^2 = tr(A_{rse}^2) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2 = 2\sum_{i\sim j} \frac{1}{e(v_i)+e(v_j)}. \tag{2}$$

In section 2 of this paper we obtain the upper and lower bounds for Randić sum eccentricity energy of graphs. In Section 3 we compute the Randić sum eccentricity energy of some graphs such as complete graph, star graph, complete bipartite graph, $(S_m \wedge P_2)$ graph and crown graph..

2 Upper and lower bounds for Randić sum eccentricity energy

Theorem 2.1. Let G be a simple graph of order n with no isolated vertex. Then

$$E_{rse}(G) \leq \sqrt{2n \sum_{j\sim k} \frac{1}{e(v_i)+e(v_j)}}.$$

Proof: Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq ... \geq \lambda_n$ be the eigenvalues of A_{rse}. Then using (2) and the Cauchy- Schwartz inequality, we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right) \cdot \left(\sum_{i=1}^{n} b_i^2\right)$$

with $a_i = 1$, $b_i =\mid \lambda_i \mid$, we obtain

$$E_{rse}(G) = \sum_{i=1}^{n} |\lambda_i| = \sqrt{(\sum_{i=1}^{n} |\lambda_i|)^2} \leq \sqrt{n \sum_{i=1}^{n} \lambda_i^2} = \sqrt{2n \sum_{i\sim j} \frac{1}{e(v_i)+e(v_j)}}.$$

∎

Theorem 2.2. Let G be a simple graph of order n with no isolated vertices. Then

$$E_{rse}(G) \geq 2\sqrt{\sum_{i\sim j} \frac{1}{e(v_i)+e(v_j)}}. \tag{4}$$

Proof: From (1), we have

$$\sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1\leq i<j\leq n} \lambda_i \lambda_j = 0$$

and therefore

$$-\sum_{i=1}^{n}\lambda_i^2 = 2\sum_{1\le i<j\le n}\lambda_i\lambda_j. \tag{5}$$

Thus

$$(E_{rse}(G))^2 = \left(\sum_{i=1}^{n}|\lambda_i|\right)^2 = \sum_{i=1}^{n}\lambda_i^2 + 2\sum_{1\le i<j\le n}|\lambda_i\lambda_j|$$

$$\ge \sum_{i=1}^{n}\lambda_i^2 + 2\left|\sum_{1\le i<j\le n}\lambda_i\lambda_j\right|$$

$$= 2\sum_{i=1}^{n}\lambda_i^2, \quad on \ \ using \ (5).$$

This together with (2) implies that

$$(E_{rse}(G))^2 \ge 4\sum_{i\sim j}\frac{1}{e(v_i)+e(v_j)},$$

which gives (4). ∎

3 Randić sum eccentricity energies of some families of graphs

We recall that the complete graph is one in which every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n. A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$[3].

The following definitions and notations, will be used in the remainder of this paper.

Definition 3.1. [2] The Crown graph S_n^0 for an integer $n \ge 3$ is the graph which is obtained from the complete bipartite graph $K_{n,n}$ by removing the horizontal edges.

Definition 3.2. [4] The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) : v_iv_k \in E(S_m)\ and\ v_jv_l \in E(P_2)\ and\ 1 \le i,k \le m+1, 1 \le j,l \le 2\}$. In fact S_m is the star $K_{1,m}$ and P_2 is K_2.

Now we compute Randić sum eccentricity energies of complete graph, star graph, complete bipartite graph, $(S_m \wedge P_2)$ graph and crown graph.

Theorem 3.3. The Randić sum eccentricity energy of complete bipartite graph $K_{m,n}$ is $\sqrt{mn}$.

Proof: Let the vertex set of the complete bipartite graph be $V(K_{m,n}) = \{u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n\}$. Then Randić sum eccentricity matrix of complete bipartite graph is given by

$$A_{rse} = \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & \cdots & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & \cdots & \frac{1}{2} & 0 & \cdots & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{rse}| = \begin{vmatrix} \lambda I_m & -\frac{1}{2}J^T \\ -\frac{1}{2}J & \lambda I_n \end{vmatrix},$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_m & -\frac{1}{2}J^T \\ -\frac{1}{2}J & \lambda I_n \end{vmatrix} = 0,$$

which can be written as

$$|\lambda I_m| \left| \lambda I_n - \left(-\frac{1}{2}J\right) \frac{I_m}{\lambda} \left(-\frac{1}{2}J^T\right) \right| = 0.$$

On simplification, we obtain

$$\frac{\lambda^{m-n}}{(4)^n} \left| (4)\lambda^2 I_n - JJ^T \right| = 0,$$

which can be written as

$$\frac{\lambda^{m-n}}{(4)^n} P_{JJ^T}(4\lambda^2) = 0,$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix JJ^T. Thus, we have

$$\frac{\lambda^{m-n}}{(4)^n}(4\lambda^2 - mn)(4\lambda^2)^{n-1} = 0,$$

which is same as

$$\lambda^{m+n-2}(\lambda^2 - \frac{mn}{4}) = 0.$$

Therefore, the spectrum of $K_{m,n}$ is given by

$$Spec\,(K_{m,n}) = \begin{pmatrix} 0 & \sqrt{\frac{mn}{4}} & -\sqrt{\frac{mn}{4}} \\ m+n-2 & 1 & 1 \end{pmatrix}.$$

Hence the Randić sum eccentricity energy of complete bipartite graph is

$$E_{rse}(K_{m,n}) = \sqrt{mn},$$

as desired. ∎

Theorem 3.4. The Randić sum eccentricity energy of $S_n (n \geq 3)$ is $2\sqrt{\frac{n-1}{3}}$.

Proof: Let the vertex set of star graph be given by $V(S_n) = \{v_1, v_2, \ldots, v_n\}$. Then the Randić sum eccentricity matrix of star graph S_n is given by

$$A_{rse} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \cdots & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{rse}| = \begin{vmatrix} \lambda & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \cdots & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \lambda & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{3}} & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \cdots & \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{3}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{3}$. Then $|\lambda I - A_{rse}| = \phi_n(\mu)\left(\frac{1}{\sqrt{3}}\right)^n$, where $\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & 0 & \cdots & 0 & 0 \\ -1 & 0 & \mu & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & \mu & 0 \\ -1 & 0 & 0 & \cdots & 0 & \mu \end{vmatrix}$.

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\mu) = (\mu\phi_{n-1}(\mu) - \mu^{n-2}).$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1)).$$

Therefore

$$|\lambda I - A_{rse}| = \left(\frac{1}{\sqrt{3}}\right)^n \left[\left((3)\lambda^2 - (n-1)\right)(\lambda\sqrt{3})^{n-2}\right].$$

Thus the characteristic equation is given by

$$\lambda^{n-2}\left(\lambda^2 - \frac{n-1}{3}\right) = 0.$$

Hence

$$Spec\,(S_n) = \begin{pmatrix} 0 & \sqrt{\frac{n-1}{3}} & -\sqrt{\frac{n-1}{3}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence the Randić sum eccentricity energy of S_n is

$$E_{rse}(S_n) = 2\sqrt{\frac{n-1}{3}}.$$

■

Theorem 3.5. The Randić sum eccentricity energy of $K_n (n \geq 2)$ is $\sqrt{2}(n-1)$.

Proof: Let the vertex set of Complete graph be given by $V(K_n) = \{v_1, v_2, \ldots, v_n\}$. Then the Randić sum eccentricity matrix of complete graph K_n is given by

$$A_{rse} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \cdots & \frac{1}{\sqrt{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{rse}| = \begin{vmatrix} \lambda & -\frac{1}{\sqrt{2}} & \cdots & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \lambda & \cdots & -\frac{1}{\sqrt{2}} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \cdots & \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix},$$

where $\mu = \lambda\sqrt{2}$. Then $|\lambda I - A_{rse}| = \phi_n(\mu)\left(\frac{1}{\sqrt{2}}\right)^n$,

where $\phi_n(\mu) = \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}$

$$= \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ 0 & 0 & 0 & \cdots & -1-\mu & \mu+1 \end{vmatrix}$$

$$= (\mu+1) \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 \end{vmatrix} + (\mu+1) \begin{vmatrix} \mu & -1 & -1 & \cdots & -1 & -1 \\ -1 & \mu & -1 & \cdots & -1 & -1 \\ -1 & -1 & \mu & \cdots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & \mu & -1 \\ -1 & -1 & -1 & \cdots & -1 & \mu \end{vmatrix}.$$

$$\phi_n(\mu) = -(\mu+1)^{n-1} + (\mu+1)\left[(\mu+1)^{n-2}(\mu-(n-2))\right]$$
$$= -(\mu+1)^{n-1} + (\mu+1)^{n-1}(\mu-(n-2)).$$

Iterating this, we obtain

$$\phi_n(\mu) = \mu^{n-2}(\mu^2 - (n-1)),$$

thus the characteristic equation is given by

$$\left(\frac{1}{\sqrt{2}}\right)^n (\mu+1)^{n-1}(\mu - (n-1)) = 0.$$

Hence

$$Spec\,(K_n) = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{n-1}{\sqrt{2}} \\ n-1 & 1 \end{pmatrix}.$$

Hence the Randić sum eccentricity energy of K_n is

$$E_{rse}(K_n) = \sqrt{2}(n-1).$$

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Theorem 3.6. The Randić sum eccentricity energy of $(S_m \wedge P_2)$ is $4\sqrt{\frac{m}{3}}$.

Proof: Let the vertex set of $(S_m \wedge P_2)$ graph be given by $V(S_m \wedge P_2) = \{v_1, v_2, \ldots, v_{2m+2}\}$. Then the Randić sum eccentricity matrix of $(S_m \wedge P_2)$ graph is given by

$$A_{rse} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{\sqrt{3}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{3}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{3}} & \cdots & 0 \\ 0 & \frac{1}{\sqrt{3}} & \cdots & \frac{1}{\sqrt{3}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{3}} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{3}} & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where $m+1=n$. Its characteristic polynomial is given by

$$|\lambda I - A_{rse}| = \begin{vmatrix} \lambda & \cdots & 0 & 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \cdots & 0 & -\frac{1}{\sqrt{3}} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda & -\frac{1}{\sqrt{3}} & \cdots & 0 \\ 0 & \cdots & -\frac{1}{\sqrt{3}} & \lambda & \cdots & 0 \\ -\frac{1}{\sqrt{3}} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{3}} & \cdots & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence the characteristic equation is given by

$$\left(\frac{1}{\sqrt{3}}\right)^{2n} \begin{vmatrix} \Lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \Lambda & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n} = 0,$$

$$where\ \Lambda = \sqrt{3}\lambda.$$

Let

$$\phi_{2n}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{2n\times 2n}.$$

$$= (-1)^{2n+2n}\Lambda \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \Lambda & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \Lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \Lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+2} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Let

$$\Psi_{2n-1}(\Lambda) = (-1)^{2n+2} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \Lambda & 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \Lambda & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \Lambda & -1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \Lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \Lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain, after some simplifications

$$\Psi_{2n-1}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda),$$

where $\Theta_n(\Lambda)$=$\begin{vmatrix} \Lambda & 0 & 0 & \cdots & -1 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{n\times n}$.

Then

$$\phi_{2n}(\Lambda) = -\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda\phi_{2n-1}(\Lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned} \phi_{2n-1}(\Lambda) &= (-1)^{(2n-1)+2}\Psi_{2n-2}(\Lambda) + (-1)^{(2n-1)+(2n-1)}\Lambda\phi_{2n-2}(\Lambda) \\ &= -\Lambda^{n-3}\Theta_n(\Lambda) + \Lambda\phi_{2n-2}(\Lambda). \end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\Lambda) = -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{(n-1)}\xi_{n+1}(\Lambda),$$

where $\xi_{n+1}(\Lambda) = \begin{vmatrix} \Lambda & 0 & 0 & \cdots & 0 \\ 0 & \Lambda & 0 & \cdots & -1 \\ 0 & 0 & \Lambda & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \Lambda \end{vmatrix}_{(n+1)\times(n+1)}$.

$$\begin{aligned} \phi_{2n}(\Lambda) &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n-1}\Lambda\Theta_n(\Lambda) \\ &= -(n-1)\Lambda^{n-2}\Theta_n(\Lambda) + \Lambda^{n}\Theta_n(\Lambda) \\ &= (\Lambda^n - (n-1)\Lambda^{n-2})\Theta_n(\Lambda). \end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\Lambda) = \Lambda^n - (n-1)\Lambda^{n-2}.$$

Therefore

$$\phi_{2n}(\Lambda) = (\Lambda^n - (n-1)\Lambda^{n-2})^2.$$

Hence characteristic equation becomes

$$\left(\frac{1}{\sqrt{3}}\right)^{2n} \phi_{2n}(\Lambda) = 0,$$

which is same as

$$\left(\frac{1}{\sqrt{3}}\right)^{2n} (\Lambda^n - (n-1)\Lambda^{n-2})^2 = 0.$$

This reduces to

$$\lambda^{2n-4}((3)\lambda^2 - (n-1))^2 = 0.$$

Therefore

$$Spec\,((S_m \wedge P_2)) = \begin{pmatrix} 0 & \sqrt{\frac{(n-1)}{3}} & -\sqrt{\frac{(n-1)}{3}} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the Randić sum eccentricity energy of $(S_m \wedge P_2)$ graph is

$$E_{rse}((S_m \wedge P_2)) = 4\sqrt{\frac{m}{3}}.$$

∎

Theorem 3.7. The Randić sum eccentricity energy of crown graph S_n^0 is $2(n-1)$.

Proof: The vertex set of the crown graph be given by $V(S_n^0) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$. Then Randić sum eccentricity matrix of crown graph is given by

$$A_{rse} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & X & \cdots & X \\ 0 & 0 & \cdots & 0 & X & 0 & \cdots & X \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & X & X & \cdots & 0 \\ 0 & X & \cdots & X & 0 & 0 & \cdots & 0 \\ X & 0 & \cdots & X & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X & X & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Where $X = \frac{1}{\sqrt{4}}$. Its characteristic polynomial is

$$|\lambda I - A_{rse}| = \begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{4}}K^T \\ -\frac{1}{\sqrt{4}}K & \lambda I_n \end{vmatrix},$$

where K is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{4}}K^T \\ -\frac{1}{\sqrt{4}}K & \lambda I_n. \end{vmatrix} = 0.$$

This is same as

$$|\lambda I_n| \left| \lambda I_n - \left(-\frac{K}{\sqrt{4}}\right) \frac{I_n}{\lambda} \left(-\frac{K^T}{\sqrt{4}}\right) \right| = 0,$$

which can be written as

$$\frac{1}{(4)^n} P_{KK^T}((4)\lambda^2) = 0,$$

where $P_{KK^T}(\lambda)$ is the characteristic polynomial of the matrix KK^T. Thus we have

$$\frac{1}{(4)^n}[4\lambda^2 - (n-1)^2][4\lambda^2 - 1]^{n-1} = 0,$$

which is same as

$$\left(\lambda^2 - \frac{(n-1)^2}{4}\right)\left(\lambda^2 - \frac{1}{4}\right)^{n-1} = 0.$$

Therefore

$$Spec\,(S_n^0) = \begin{pmatrix} \sqrt{\frac{(n-1)^2}{4}} & -\sqrt{\frac{(n-1)^2}{4}} & \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Hence the Randić sum-eccentricity energy of crown graph is

$$E_{rse}(S_n^0) = 2(n-1),$$

as desired. ∎

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