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On Skew-Sum-Connectivity Energy of Digraphs

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Abstract

In this paper we define the skew-sum-connectivity energy of a digraph and we obtain upper and lower bounds for skew-sum-connectivity energy of digraphs. Also we compute the skew-sum-connectivity energy of star digraph, complete bipartite digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph, with respect to the indegree and outdegree of the vertices.

Key words: Skew-sum-connectivity energy

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1 Introduction

In 2010, Bo Zhou and Nenad Trinajstic [3] have introduced the sum-connectivity energy of a graph as follows. Let G be a simple graph and let $v_1, v_2, \ldots, v_n$ be its vertices. For $i = 1, 2, \ldots, n$, let d_i denote the degree of the vertex v_i, where the degree of the vertex v_i is the number of vertices adjacent to it. Then the sum-connectivity matrix of G is defined as $R = (R_{ij})$, where

$$R_{ij} = \begin{cases} 0, & \text{if } i = j, \\ \frac{1}{\sqrt{d_i + d_j}}, & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{if the vertices } v_i \text{ and } v_j \text{ are not adjacent.} \end{cases}$$

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The sum-connectivity energy of G is defined as the sum of absolute values of the eigenvalues of the sum-connectivity matrix of G.

In the same year Adiga et.al [1] have introduced the skew energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. The skew-adjacency matrix of D is the $n \times n$ matrix $S(D) = (s_{ij})$ where $s_{ij} = 1$ whenever $(v_i, v_j) \in \Gamma(D)$, $s_{ij} = -1$ whenever $(v_j, v_i) \in \Gamma(D)$ and $s_{ij} = 0$ otherwise. Hence $S(D)$ is a skew symmetric matrix of order n and all its eigenvalues are of the form $i\lambda$ where $i = \sqrt{-1}$ and λ is a real number. The skew energy of G is the sum of the absolute values of eigenvalues of $S(D)$.

Motivated by these works, we introduce the concept of skew-sum-connectivity energy of a digraph as follows. Let D be a digraph of order n with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and arc set $\Gamma(D) \subset V(D) \times V(D)$ where $(v_i, v_i) \notin \Gamma(D)$ for all i and $(v_i, v_j) \in \Gamma(D)$ implies $(v_j, v_i) \notin \Gamma(D)$. If $(v_i, v_j)$is an arc, we say that v_i is adjacent to v_j and v_j is adjacent from v_i. The indegree $id(v)$ of a vertex v is the number of vertices adjacent to it and the outdegree $od(v)$ of a vertex v is the number of vertices adjacent from it [4]. Let d_i denote the indegree of the vertex v_i, for $1 \leq i \leq n$. Then the skew-sum-connectivity matrix of D is the $n \times n$ matrix $A_{SSC} = (a_{ij})$ where

$$a_{ij} = \begin{cases} \frac{1}{\sqrt{d_i+d_j}}, & \text{if } (v_i, v_j) \in \Gamma(D), \\ -\frac{1}{\sqrt{d_i+d_j}}, & \text{if } (v_j, v_i) \in \Gamma(D), \\ 0, & \text{otherwise.} \end{cases}$$

Then the skew-sum-connectivity energy of D with respect to the indegree of the vertices is denoted by $E_{ssc}(D)$ and is defined as the sum of the absolute values of eigenvalues of A_{SSC}.

For example let D be the directed cycle on 4 vertices with the arc set $\{(1,2), (2,3), (3,4), (4,1)\}$. Then

$$A_{SSC} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}.$$

Then the characteristic equation is given by $\lambda^4 + 2\lambda^2 = 0$. The eigenvalues are $\sqrt{2}i$, 0, 0, $-\sqrt{2}i$ and skew-sum-connectivity energy of D is $2\sqrt{2}$. The skew-sum-connectivity energy of

D with respect to the outdegree of the vertices can be defined in the same manner.

In section 2 of this paper we obtain the upper and lower bounds for skew-sum-connectivity energy of digraphs. In Section 3, We compute the skew-sum-connectivity energy of star digraph, complete bipartite digraph, the $(S_m \wedge P_2)$ digraph, $(n, 2n-3)$ strong vertex graceful digraph and a crown digraph with respect to indegree of the vertices. In section 4 we give the table which shows the skew-sum-connectivity energies with respect to the indegree and outdegree of the vertices.

2 Upper and lower bounds for skew-sum-connectivity energy

Theorem 2.1. Let D be a simple digraph of order n. Then

$$E_{SSC}(D) \leq \sqrt{2n \sum_{j \sim k} \frac{1}{d_j + d_k}}.$$

Proof: Let $i\lambda_1, i\lambda_2, i\lambda_3, ..., i\lambda_n$, be the eigenvalues of A_{SSC}, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq ... \geq \lambda_n$. Since

$$\sum_{j=1}^{n} (i\lambda_j)^2 = tr(A_{SSC}^2) = -\sum_{j=1}^{n} \sum_{k=1}^{n} a_{jk}^2 = -2 \sum_{j \sim k} \frac{1}{d_j + d_k},$$

we have

$$\sum_{j=1}^{n} |\lambda_j|^2 = 2 \sum_{j \sim k} \frac{1}{d_j + d_k}. \tag{1}$$

Applying the Cauchy-Schwartz inequality

$$\left(\sum_{j=1}^{n} a_j b_j \right)^2 \leq \left(\sum_{j=1}^{n} a_j^2 \right) \cdot \left(\sum_{j=1}^{n} b_j^2 \right)$$

with $a_j = 1$, $b_j = \mid \lambda_j \mid$, we obtain

$$E_{SSC}(D) = \sum_{j=1}^{n} |\lambda_j| = \sqrt{(\sum_{j=1}^{n} |\lambda_j|)^2} \leq \sqrt{n \sum_{j=1}^{n} |\lambda_j|^2} = \sqrt{2n \sum_{j \sim k} \frac{1}{d_j + d_k}}.$$

∎

Theorem 2.2. Let D be a simple digraph of order n. Then

$$E_{SSC}(D) \geq \sqrt{2\sum_{j\sim k}\frac{1}{d_j+d_k}+n(n-1)p^{\frac{2}{n}}},\ where\ \ p=|detA_{SSC}|=\prod_{j=1}^{n}|\lambda_j|. \tag{2}$$

Proof:

$$(E_{SSC}(D))^2=\left(\sum_{j=1}^{n}|\lambda_j|\right)^2=\sum_{j=1}^{n}|\lambda_j|^2+\sum_{1\leq j\neq k\leq n}|\lambda_j||\lambda_k|.$$

By arithmetic $-$ geometric mean inequality, we get

$$\begin{aligned}
&\sum_{1\leq j\neq k\leq n}|\lambda_j||\lambda_k| \\
&\geq n(n-1)(|\lambda_1||\lambda_2|...|\lambda_n|)^{\frac{1}{n}}(|\lambda_1|^{n-1}|\lambda_2|^{n-1}...|\lambda_n|^{n-1})^{\frac{1}{n(n-1)}} \\
&= n(n-1)(\prod_{j=1}^{n}|\lambda_j|)^{\frac{1}{n}}(\prod_{j=1}^{n}|\lambda_j|)^{\frac{1}{n}} \\
&= n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}}.
\end{aligned}$$

Thus

$$(E_{SSC}(D))^2\geq\sum_{j=1}^{n}|\lambda_j|^2+n(n-1)\left(\prod_{j=1}^{n}|\lambda_j|\right)^{\frac{2}{n}}.$$

From the equation (1), we get

$$(E_{SSC}(D))^2\geq 2\sum_{j\sim k}\frac{1}{d_j+d_k}+n(n-1)p^{\frac{2}{n}},$$

which gives (2). ∎

3 Skew- sum-connectivity energies of some families of graphs

We begin with some basic definitions and notations.

Definition 3.1. [4] A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.

Definition 3.2. [4] A bigraph or bipartite graph G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. (V_1, V_2) is a bipartition of G. If G contains every line joining V_1 and V_2, then G is a complete bigraph. If V_1 and V_2 have m and n points, we write $G = K_{m,n}$. A star is a complete bigraph $K_{1,n}$.

Definition 3.3. [2] The Crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{u_i v_j; 1 \leq i, j \leq n, i \neq j\}$. S_n^0 is therefore equivalent to the complete bipartite graph $K_{n,n}$ from which the edges of perfect matching have been removed.

Definition 3.4. [5] The conjunction $(S_m \wedge P_2)$ of $S_m = \overline{K}_m + K_1$ and P_2 is the graph having the vertex set $V(S_m) \times V(P_2)$ and edge set $\{(v_i, v_j)(v_k, v_l) | v_i v_k \in E(S_m)\ and\ v_j v_l \in E(P_2)\ and\ 1 \leq i, k \leq m+1, 1 \leq j, l \leq 2\}$.

Definition 3.5. [6] A graph G is said to be strong vertex graceful if there exists a bijective mapping $f : V(G) \to \{1, 2, \ldots, n\}$ such that for the induced labeling $f^+ : E(G) \to \mathbb{N}$ defined by $f^+(e) = f(u) + f(v)$, where $e = uv$, the set $f^+(E(G))$ consists of consecutive integers.

Theorem 3.6. Let the vertex set $V(K_{m,n})$ and arc set $\Gamma(K_{m,n})$ of the complete bipartite digraph be respectively given by

$$V(K_{m,n}) = \{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\} \text{ and}$$
$$\Gamma(K_{m,n}) = \{(u_i, v_j) \,|\, 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Then the skew-sum-connectivity energy of the complete bipartite digraph is $2\sqrt{n}$.

Proof: The skew-sum-connectivity matrix of complete bipartite digraph is given by

$$A_{SSC} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} \\ -\frac{1}{\sqrt{m}} & -\frac{1}{\sqrt{m}} & \cdots & -\frac{1}{\sqrt{m}} & 0 & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{m}} & -\frac{1}{\sqrt{m}} & \cdots & -\frac{1}{\sqrt{m}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{m}} & -\frac{1}{\sqrt{m}} & \cdots & -\frac{1}{\sqrt{m}} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$
\begin{aligned}
|\lambda I - A_{SSC}| &= \begin{vmatrix}
\lambda & 0 & \cdots & 0 & -\frac{1}{\sqrt{m}} & -\frac{1}{\sqrt{m}} & \cdots & -\frac{1}{\sqrt{m}} \\
0 & \lambda & \cdots & 0 & -\frac{1}{\sqrt{m}} & -\frac{1}{\sqrt{m}} & \cdots & -\frac{1}{\sqrt{m}} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda & -\frac{1}{\sqrt{m}} & -\frac{1}{\sqrt{m}} & \cdots & -\frac{1}{\sqrt{m}} \\
\frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} & \lambda & 0 & \cdots & 0 \\
\frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} & 0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{\sqrt{m}} & \frac{1}{\sqrt{m}} & \cdots & \frac{1}{\sqrt{m}} & 0 & 0 & \cdots & \lambda
\end{vmatrix} \\
&= \begin{vmatrix}
\lambda I_m & -\frac{1}{\sqrt{m}} J^T \\
\frac{1}{\sqrt{m}} J & \lambda I_n
\end{vmatrix},
\end{aligned}
$$

where J is an $n \times m$ matrix with all the entries are equal to 1. Hence the characteristic equation is given by

$$
\begin{vmatrix}
\lambda I_m & -\frac{1}{\sqrt{m}} J^T \\
\frac{1}{\sqrt{m}} J & \lambda I_n
\end{vmatrix} = 0.
$$

This is same as

$$
\begin{aligned}
|\lambda I_m| \left| \lambda I_n - \left(\frac{1}{\sqrt{m}} J \right) \frac{I_m}{\lambda} \left(-\frac{1}{\sqrt{m}} J^T \right) \right| &= 0 \\
\frac{\lambda^{m-n}}{m^n} \left| m\lambda^2 I_n + J J^T \right| &= 0 \\
\frac{\lambda^{m-n}}{m^n} P_{JJ^T}(-m\lambda^2) &= 0,
\end{aligned}
$$

where $P_{JJ^T}(\lambda)$ is the characteristic polynomial of the matrix ${}_mJ_n$. Thus, we have

$$
\frac{\lambda^{m-n}}{m^n} (m\lambda^2 + mn)(m\lambda^2)^{n-1} = 0,
$$

which is same as

$$
\lambda^{m+n-2} (\lambda^2 + \frac{mn}{m}) = 0.
$$

Therefore, the spectrum of $K_{m,n}$ is

$$
Spec\,(K_{m,n}) = \begin{pmatrix}
0 & i\sqrt{n} & -i\sqrt{n} \\
m+n-2 & 1 & 1
\end{pmatrix}.
$$

Hence the skew-sum-connectivity energy of the complete bipartite digraph is

$$E_{SSC}(K_{m,n}) = 2\sqrt{n},$$

as desired. ∎

Theorem 3.7. Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of S_n star digraph be respectively given by

$$V(D) = \{v_1, v_2, \ldots, v_n\} \text{ and } \Gamma(D) = \{(v_1, v_j) \mid 2 \leq j \leq n\}.$$

Then the skew-sum-connectivity energy of D is $2\sqrt{n-1}$.

Proof: The skew-sum-connectivity matrix of the star digraph D is given by

$$A_{SSC} = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Hence the characteristic polynomial is given by

$$|\lambda I - A_{SSC}| = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 \\ 1 & \lambda & 0 & \cdots & 0 \\ 1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & \lambda \end{vmatrix}.$$

Then $|\lambda I - A_{SSC}| = \phi_n(\lambda)$,

where $\phi_n(\lambda) = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 & -1 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 1 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \lambda & 0 \\ 1 & 0 & 0 & \cdots & 0 & \lambda \end{vmatrix}.$

Using the properties of the determinants, we obtain after some simplifications

$$\phi_n(\lambda) = (\lambda^{n-2} + \lambda\phi_{n-1}(\lambda)).$$

Iterating this, we obtain

$$\phi_n(\lambda) = \lambda^{n-2}(\lambda^2 + n - 1).$$

Therefore

$$|\lambda I - A_{SSC}| = \lambda^{n-2}(\lambda^2 + n - 1).$$

Thus the characteristic equation is given by

$$\lambda^{n-2}(\lambda^2 + n - 1) = 0.$$

Hence

$$Spec\,(D) = \begin{pmatrix} 0 & i\sqrt{n-1} & -i\sqrt{n-1} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence the skew-sum-connectivity energy of D is

$$E_{SSC}(D) = 2\sqrt{n-1}.$$

■

Remark: Theorem 3.7 can be obtained as a corollary to Theorem 3.6 by replacing m by 1 and n by $n-1$.

Theorem 3.8. Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $(S_m \wedge P_2)$ digraph be respectively given by

$$V(D) = \{v_1, v_2, \ldots, v_{2m+2}\} \text{ and}$$
$$\Gamma(D) = \{(v_1, v_j), (v_{m+2}, v_k) \,|\, 2 \leq k \leq m+1, m+3 \leq j \leq 2m+2\}.$$

Then the skew-sum-connectivity energy of D is $4\sqrt{m}$.

Proof: The skew-sum-connectivity matrix of $(S_m \wedge P_2)$ digraph is given by

$$A_{SSC} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n},$$

where $m + 1 = n$. Its characteristic polynomial is given by

$$|\lambda I - A_{SSC}| = \begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \lambda & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 \\ 0 & \lambda & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & -1 & \lambda & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n} = 0.$$

Let

$$\phi_{2n}(\lambda)=\begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{2n\times 2n}.$$

$$=(-1)^{2n+2n}\lambda\begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & \cdots & -1 \\ 0 & \lambda & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & \cdots & -1 & \lambda & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{(2n-1)\times(2n-1)}$$

$$+(-1)^{2n+1}\begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}.$$

Let

$$\Psi_{2n-1}(\lambda) = (-1)^{2n+1} \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -1 & \cdots & -1 & -1 \\ \lambda & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & -1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \lambda & 0 \end{vmatrix}_{(2n-1)\times(2n-1)}$$

Using the properties of the determinants, we obtain, after some simplifications

$$\Psi_{2n-1}(\lambda) = \lambda^{n-2}\Theta_n(\lambda),$$

where $\Theta_n(\lambda)=\begin{vmatrix} \lambda & 0 & 0 & \cdots & 1 \\ 0 & \lambda & 0 & \cdots & 1 \\ 0 & 0 & \lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & \lambda \end{vmatrix}_{n\times n}$.

Then

$$\phi_{2n}(\lambda) = \lambda^{n-2}\Theta_n(\lambda) + \lambda\phi_{2n-1}(\lambda).$$

Now, proceeding as above, we obtain

$$\begin{aligned} \phi_{2n-1}(\lambda) &= (-1)^{(2n-1)+1}\Psi_{2n-2}(\lambda) + (-1)^{(2n-1)+(2n-1)}\lambda\phi_{2n-2}(\lambda) \\ &= \lambda^{n-3}\Theta_n(\lambda) + \lambda\phi_{2n-2}(\lambda). \end{aligned}$$

Proceeding like this, we obtain at the $(n-1)^{th}$ step

$$\phi_{2n}(\lambda) = (n-1)\lambda^{n-2}\Theta_n(\lambda) + \lambda^{(n-1)}\xi_{n+1}(\lambda),$$

where $\xi_{n+1}(\lambda) = \begin{vmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 1 \\ 0 & 0 & \lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & -1 & \cdots & \lambda \end{vmatrix}_{(n+1)\times(n+1)}$.

$$\begin{aligned} \phi_{2n}(\lambda) &= (n-1)\lambda^{n-2}\Theta_n(\lambda) + \lambda^{n-1}\left(\lambda\Theta_n(\lambda)\right) \\ &= (n-1)\lambda^{n-2}\Theta_n(\lambda) + \lambda^{n}\Theta_n(\lambda) \\ &= ((n-1)\lambda^{n-2} + \lambda^{n})\Theta_n(\lambda). \end{aligned}$$

Using the properties of the determinants, we obtain

$$\Theta_n(\lambda) = (n-1)\lambda^{n-2} + \lambda^{n}.$$

Therefore

$$\phi_{2n}(\lambda) = ((n-1)\lambda^{n-2} + \lambda^{n})^2.$$

Hence characteristic equation becomes

$$\phi_{2n}(\lambda) = 0,$$

which is same as

$$((n-1)\lambda^{n-2} + \lambda^{n})^2 = 0.$$

This reduces to

$$\lambda^{2n-4}(n-1+\lambda^2)^2 = 0.$$

Therefore

$$Spec\left((S_m \wedge P_2)\right) = \begin{pmatrix} 0 & i\sqrt{n-1} & -i\sqrt{n-1} \\ 2n-4 & 2 & 2 \end{pmatrix}.$$

Hence the skew-sum-connectivity energy of $(S_m \wedge P_2)$ digraph is

$$E_{SSC}((S_m \wedge P_2)) = 4\sqrt{m}.$$

∎

Theorem 3.9. Let the vertex set $V(S_n^0)$ and arc set $\Gamma(S_n^0)$ of the crown digraph be respectively given by

$$V(S_n^0) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\} \text{ and}$$

$$\Gamma(S_n^0) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq n, i \neq j\}.$$

Then the skew-sum-connectivity energy of the crown digraph is $4\sqrt{n-1}$.

Proof: The skew-sum-connectivity matrix of crown digraph is given by

$$A_{SSC} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{1}{\sqrt{n-1}} & \cdots & \frac{1}{\sqrt{n-1}} \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n-1}} & 0 & \cdots & \frac{1}{\sqrt{n-1}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n-1}} & \cdots & 0 \\ 0 & -\frac{1}{\sqrt{n-1}} & \cdots & -\frac{1}{\sqrt{n-1}} & 0 & 0 & \cdots & 0 \\ -\frac{1}{\sqrt{n-1}} & 0 & \cdots & -\frac{1}{\sqrt{n-1}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{n-1}} & -\frac{1}{\sqrt{n-1}} & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Its characteristic polynomial is

$$|\lambda I - A_{SSC}| = \begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{n-1}}K^T \\ \frac{1}{\sqrt{n-1}}K & \lambda I_n \end{vmatrix},$$

where K is an $n \times n$ matrix. Hence the characteristic equation is given by

$$\begin{vmatrix} \lambda I_n & -\frac{1}{\sqrt{n-1}}K^T \\ \frac{1}{\sqrt{n-1}}K & \lambda I_n. \end{vmatrix} = 0.$$

This is same as

$$|\lambda I_n| \left| \lambda I_n - \left(\frac{K}{\sqrt{n-1}}\right) \frac{I_n}{\lambda} \left(-\frac{K^T}{\sqrt{n-1}}\right) \right| = 0,$$

which can be written as

$$\frac{1}{(n-1)^n} P_{KK^T}(-(n-1)\lambda^2) = 0,$$

where $P_{KK^T(\lambda)}$ is the characteristic polynomial of the matrix KK^T. Thus we have

$$\frac{1}{(n-1)^n}[(n-1)\lambda^2 + (n-1)^2][(n-1)\lambda^2 + 1]^{n-1} = 0,$$

which is same as

$$\left(\lambda^2 + n - 1\right)\left(\lambda^2 + \frac{1}{(n-1)}\right)^{n-1} = 0.$$

Therefore

$$Spec\,(S_n^0) = \begin{pmatrix} i\sqrt{n-1} & -i\sqrt{n-1} & \frac{i}{\sqrt{n-1}} & -\frac{i}{\sqrt{n-1}} \\ 1 & 1 & n-1 & n-1 \end{pmatrix}.$$

Hence the skew-sum-connectivity energy of crown digraph is

$$E_{SSC}(S_n^0) = 4\sqrt{n-1},$$

as desired. ∎

Theorem 3.10. Let the vertex set $V(D)$ and arc set $\Gamma(D)$ of $(n, 2n-3)$ strong vertex graceful digraph $D = K_2 + \overline{K}_{n-2}$ be respectively given by

$$V(D) = \{v_1, v_2, \ldots, v_n\} \text{ and } \Gamma(D) = \{(v_1, v_j)\,|\,2 \le j \le n\} \cup \{(v_j, v_n)\,|\,2 \le j \le n-1\}.$$

Then the skew-sum-connectivity energy of D is $2\sqrt{\frac{n^3-2n^2+2}{n(n-1)}}$.

Proof: The skew-sum-connectivity matrix is given by

$$A_{SSC} = \begin{pmatrix} 0 & 1 & 1 & \cdots & \frac{1}{\sqrt{n-1}} \\ -1 & 0 & 0 & \cdots & \frac{1}{\sqrt{n}} \\ -1 & 0 & 0 & \cdots & \frac{1}{\sqrt{n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{\sqrt{n-1}} & -\frac{1}{\sqrt{n}} & -\frac{1}{\sqrt{n}} & \cdots & 0 \end{pmatrix}.$$

Its characteristics polynomial is

$$\begin{aligned} |\lambda I - A_{SSC}| &= \begin{vmatrix} \lambda & -1 & -1 & \cdots & -\frac{1}{\sqrt{n-1}} \\ 1 & \lambda & 0 & \cdots & -\frac{1}{\sqrt{n}} \\ 1 & 0 & \lambda & \cdots & -\frac{1}{\sqrt{n}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{n-1}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \lambda \end{vmatrix} \\ &= \frac{1}{n} \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 & -\sqrt{\frac{n}{n-1}} \\ 1 & \lambda & 0 & \cdots & 0 & -1 \\ 1 & 0 & \lambda & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & \lambda & -1 \\ \sqrt{\frac{n}{n-1}} & 1 & 1 & \cdots & 1 & \lambda \end{vmatrix}. \end{aligned}$$

Using the properties of the determinants, we obtain, after some simplifications

$$|\lambda I - A_{SSC}| = \frac{1}{n}\left[(-1)^{2n+1}\left(\lambda^2 - \frac{n}{n-1}\right)\lambda^{n-2} + (-1)^{2n}(n+1)\lambda\phi_{n-1}(\lambda)\right], \qquad (1)$$

where $\phi_{n-1}(\lambda) = \begin{vmatrix} \lambda & -1 & -1 & \cdots & -1 \\ 1 & \lambda & 0 & \cdots & 0 \\ 1 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & \lambda \end{vmatrix}_{(n-1)\times(n-1)}$.

Now, as in the proof of the Theorem 2.2, we obtain

$$\phi_{n-1}(\lambda) = \lambda^{n-3} + \lambda\phi_{n-2}(\lambda).$$

Iterating this, we obtain

$$\phi_{n-1}(\lambda) = \lambda^{n-3}(\lambda^2 + n - 2). \qquad (2)$$

Substituting (2) in (1), we obtain

$$\begin{aligned} |\lambda I - A_{SSC}| &= \frac{1}{n}\left[-\left(\lambda^2 - \frac{n}{n-1}\right)\lambda^{n-2} + (n+1)\lambda^{n-2}(\lambda^2 + n - 2)\right] \\ &= \frac{\lambda^{n-2}}{n}\left(n\lambda^2 + \frac{n}{n-1} + (n+1)(n-2)\right). \end{aligned}$$

Thus the characteristic equation is given by

$$\lambda^{n-2}\left(\lambda^2 + \frac{n^3 - 2n^2 + 2}{n(n-1)}\right) = 0.$$

Hence

$$Spec\,(D) = \begin{pmatrix} 0 & i\sqrt{\frac{n^3-2n^2+2}{n(n-1)}} & -i\sqrt{\frac{n^3-2n^2+2}{n(n-1)}} \\ n-2 & 1 & 1 \end{pmatrix}.$$

Hence the skew-sum-connectivity energy of D is

$$E_{SSC}(D) = 2\sqrt{\frac{n^3 - 2n^2 + 2}{n(n-1)}},$$

as desired. ∎

4 The comparison of skew-sum-connectivity energies.

We remark that the skew-sum-connectivity energy of a digraph varies with types of degrees of the vertices. We give below the table showing the skew-sum-connectivity energy of the digraphs with respect to the indegree and outdegree of the vertices.

Digraph	$E_{ssc}(D)$ with respect to the indegree of the vertices	$E_{ssc}(D)$ with respect to the outdegree of the vertices
$K_{m,n}$	$2\sqrt{n}$	$2\sqrt{m}$
S_n	$2\sqrt{n-1}$	2
$(S_m \wedge P_2)$	$4\sqrt{m}$	4
S_n^0	$4\sqrt{n-1}$	$4\sqrt{n-1}$
$K_2 + \overline{K}_{n-2}$	$2\sqrt{\frac{n^3-2n^2+2}{n^2-n}}$	$2\sqrt{\frac{n^3-2n^2+2}{n^2-n}}$

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