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On The Hermitian Estrada index of Mixed Graphs

A. Jahanbani

Department of Mathematics
Shahrood University of Technology, Shahrood, Iran
Akbar.jahanbani92@gmail.com

Abstract

Let M be a mixed graph of order n and size m. The Hermitian-adjacency matrix is define $H(M) = (h)pq$ of a mixed graph M, where i if (p, q) is an arc of M, $-i$ if (q, p) is an arc of M, 1 if pq is an edge of M and $(h)pq = 0$ otherwise. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be its eigenvalues of the Hermitian matrix. The Hermitian energy of a mixed graph M, is defined as the sum of the absolute values of all eigenvalues the Hermitian matrix. The main purposes of this paper are to introduce the Hermitian Estrada index of a graph. We also obtain upper and lower bounds for the Hermitian Estrada index. Finally, we investigate the relations between the Hermitian Estrada index and the Hermitian energy.

Key words: Mixed graphs, Hermitian Estrada index, Hermitian Energy.

2010 Mathematics Subject Classification : 05C40, 05C69

1 Introduction

In this paper, we only consider simple graphs without multi edges and loops. A graph M is said to be mixed if it is obtained from an undirected graph M_U by orienting a subset of its edges. We call M_U the underlying graph of M. Clearly, a mixed graph concludes both possibilities of all edges oriented and all edges undirected as extreme cases. Let M be a mixed graph with vertex set $V(M) = \{v_1, v_2, ..., v_n\}$ and edge set $E(M)$. For $v_p, v_q \in V(M)$, we denote an undirected edge joining two vertices v_p and v_q of M by $v_p v_q$ (or $v_p \leftrightarrow v_q$). Denote a directed edge (or arc) from v_p to v_q by (v_p, v_q) (or $v_p \rightarrow v_q$). For undefined terminology and notation, we refer the reader to Ref.[4]. Let G be a mixed graph, then the mixed adjacency matrix M is defined entrywise as [1]

$$M(m)_{pq} = \begin{cases} 1 & \text{if } (p, q) \text{ is an edge,} \\ 1 & \text{if } (p, p) \text{ is an arc,} \\ -1 & \text{if } (q, p) \text{ is an arc,} \\ 0 & \text{otherwise.} \end{cases}$$

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The spectrum of $M(m)$ is defined throughout as $Sp(M) = (\lambda_1, \lambda_2, ..., \lambda_n)$, where $\lambda_i(M)(i = 1, 2, ..., n)$ is an eigenvalue of $H(M)$. Obviously, $M(m) = M(m)^* := \overline{M(m)}^T$. Thus all its eigenvalues are real. The concept of the energy of an undirected graph G was introduced by Ivan Gutman [12] and is defined to be the sum of the absolute values of all the eigenvalues of the adjacency matrix of G. The notion of skew-adjacency matrix and skew-energy of a digraph was introduced by Adiga, Balakrishnan and So [2]. For more information about the energy of an undirected graph, the reader may refer to [3, 5, 17, 18, 19] and therein references. Consequently, the energy of the mixed graph M, denoted by $E\,H(M)$, which is defined as the sum of its singular values [1], is also the sum of the absolute values of its eigenvalues. Recently J. Liu and X. Li [26] introduced the Hermitian adjacency matrix of a mixed graph M of order n, which is denoted by $H(M)$ and is defined as follows:

$$H(M) = (h_{pq})n \times n = \begin{cases} 1 & \text{if } v_p v_q \text{ is an edge,} \\ i & \text{if } (v_p, v_q) \text{ is an arc,} \\ -i & \text{if } (v_q, v_p) \text{ is an arc,} \\ 0 & \text{otherwise.} \end{cases}$$

Here $i = \sqrt{1}$. Note that $H(M) = A(G_1) + iS(G_2^\sigma)$ where $A(G_1)$ is the adjacency matrix of the undirected graph G_1 and $S(G_2^\sigma)$ is the skew-adjacency matrix of the digraph G_2^σ. Hence $H(M)$ is a complex Hermitian matrix, and so its eigenvalues are always real. The Hermitian adjacency matrix incorporates both adjacency matrix of an undirected graph and skew-adjacency matrix of a digraph. The spectrum $S_{p_H}(M)$ of M is defined as the spectrum of $H(M)$. It is easy to see that $H(M)$ is a Hermitian matrix, in other words its conjugation and transposition is itself, that is $H = H^* := H^T$. Thus all its eigenvalues $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ are real, and the singular values of $H(M)$ coincide with the absolute values $\{|\alpha_1|, |\alpha_2|, ..., |\alpha_n|\}$ of its eigenvalues.

The Hermitian energy of M is then defined as [26]

$$E_h = E_h(M) = \sum_{i=1}^{n} \mid \alpha_i \mid .$$

For more details about the Hermitian-adjacency matrix and the Hermitian energy of mixed graphs, we can refer to [13, 26, 27]. The adjacency matrix of an undirected graph G of ordernis the $n \times n$ matrix $A(G) = (a_{pq})$, where $a_{pq} = a_{qp} = 1$ if $v_p \sim v_q$ or $(v_p v_q \in E(M))$ and $a_{pq} = 0$

otherwise. The spectrum $S_{p_A}(G)$ of G is defined as the spectrum of $A(G)$. Since $A(G)$ is symmetric matrix, all its eigenvalues, denoted by $\{\rho_1, \rho_2, ..., \rho_n\}$, are real. The concept of the Estrada index of an undirected graph G was introduced by Estrada [10, 11] is defined as

$$EE = EE(G) = \sum_{i=1}^{n} e^{\rho_i}.$$

EE is nowadays usually referred to as the Estrada index, see [25]. Although invented only a few years ago [10], the Estrada index has already found numerous applications. It was used to quantify the degree of folding of long-chain molecules, especially proteins [11]. Some mathematical properties of the Estrada index were established in [6, 14, 15, 16, 20, 21, 22, 23, 24]. One of most important properties is the following:

$$EE = \sum_{k=1}^{\infty} \frac{M_k(G)}{k!}.$$

Denoting by $M_k = M_k(G)$ to the k-$th\ moment$ of the graph G, we get where, $M_k = M_k(G) = \sum_{i=1}^{n}(\rho_i)^k$. It is well known that $M_k(G)$ is equal to the number of closed walks of length k in G. Let thus M be a mixed graph of order n whose the Hermitian matrix eigenvalues are $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$. Then the Hermitian Estrada index of M, denoted by $EE_h(M)$, is defined to be

$$EE_h = EE_h(M) = \sum_{i=1}^{n} e^{\alpha_i}.$$

Also

$$N_k = \sum_{i=1}^{n}(\alpha_i)^k.$$

Then

$$EE_h(M) = \sum_{i=1}^{\infty} \frac{N_k}{k!}.$$

This paper is organized as follows. In the Section 2, we give a list of some previously known results. In the Section 3, introducing the Hermitian Estrada index and we establish upper and lower bounds for it. In the Section 4, we investigate the relations between the Hermitian Estrada index and the Hermitian energy.

2 Preliminaries and known results

In this section, we shall list some previously known results that will be needed in the next sections. We obtain upper bound for $\sum_{i=1}^{n} (\alpha_i)^4$.

Now we give some lemmas which will be needed then.

Lemma 2.1. [26] Let M be a mixed graph with the Hermitian matrix $H(M)$, then the following are equivalent:

$$(1) \quad N_1 = \sum_{i=1}^{n} \alpha_i = tr(M) = 0, \tag{1}$$

$$(2) \quad N_2 = \sum_{i=1}^{n} (\alpha_i)^2 = tr(M^2) = 2m. \tag{2}$$

Lemma 2.2. Let M be a mixed graph of order n, size m and $\alpha_1, \alpha_2, ..., \alpha_n$ be the Hermitian spectrum of $H(M)$.Then, the following inequality is valid

$$\sum_{i=1}^{n} \alpha_i^4 \leqslant 2m(\alpha_n^2 + \alpha_1^2) - n\alpha_1^2\alpha_n^2. \tag{3}$$

Proof: Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers for which there exist real constants R and r, so that for each i, $i = 1, 2, \ldots, n$, holds $ra_i \leqslant b_i \leqslant Ra_i$. Then the following inequality is valid (see [7])

$$\sum_{i=1}^{n} b_i^2 + rR\sum_{i=1}^{n} a_i^2 \leqslant (r+R)\sum_{i=1}^{n} a_i b_i. \tag{4}$$

Equality in (4) holds if and only if for et least one $i, 1 \leqslant i \leqslant n$ holds $ra_i = bi = Ra_i$.
For $a_i := 1$, $b_i := \alpha_i^2$, $r := \alpha_n^2$ and $R := \alpha_1^2$ for $i = 1, 2, \ldots, n$, inequality (4) becomes

$$\sum_{i=1}^{n} \alpha_i^4 + \alpha_1^2\alpha_n^2 \sum_{i=1}^{n} 1 \leqslant (\alpha_n^2 + \alpha_1^2) \sum_{i=1}^{n} \alpha_i^2.$$

By Equalitity (2), we have

$$\sum_{i=1}^{n} \alpha_i^2 = 2m.$$

Also, we know that

$$\sum_{i=1}^{n} 1 = n.$$

Therefore, the above inequality becomes

$$\sum_{i=1}^{n} \alpha_i^4 \leqslant 2m(\alpha_n^2 + \alpha_1^2) - n\alpha_1^2\alpha_n^2.$$

If for some i holds that $ra_i = b_i = Ra_i$, then for the same i also holds $b_i = r = R$. ■

Remark 2.3. For any real x, the $power\text{-}series$ expansion of e^x, is the following

$$e^x = \sum_{k \geqslant 0} \frac{x^k}{k!}. \tag{5}$$

Remark 2.4. For nonnegative $x_1, x_2, \dots, x_n$ and $k \geqslant 2$,

$$\sum_{i=1}^{n} (x_i)^k \leqslant (\sum_{i=1}^{n} {x_i}^2)^{\frac{k}{2}}. \tag{6}$$

3 Bounds for the Hermitian Estrada index of mixed graphs

In this section, we consider the Hermitian Estrada index of mixed graph G. Also we present lower and upper bounds for the Hermitian Estrada index in terms of the number of vertices and the edges of mixed graph M.
At first, we state the following theorem.

Theorem 3.1. Let M be a mixed graph of order n and size m, then

$$EE_h(M) \leqslant n - 1 + e^{\sqrt{2m-1}}. \tag{7}$$

Proof: Let the number of positive eigenvalues of G be n_+. Since $f(x) = e^x$ monotonically increases in the interval $(-\infty, +\infty)$ and $m \neq 0$, we get

$$\begin{aligned} EE_h(M) = \sum_{i=1}^{n} e^{\alpha_i} < n - n_+ + \sum_{i=1}^{n_+} e^{\alpha_i} &= n - n_+ + \sum_{i=1}^{n_+} \sum_{k \geqslant 0} \frac{(\alpha_i)^k}{k!} \\ &= n + \sum_{k \geqslant 1} \frac{1}{k!} \sum_{i=1}^{n_+} (\alpha_i)^k \\ &\leqslant n + \sum_{k \geqslant 1} \frac{1}{k!} \Big[\sum_{i=1}^{n_+} \alpha_i^2 \Big]^{\frac{k}{2}} \end{aligned} \tag{8}$$

$$= n + \sum_{k\geqslant 1} \frac{1}{k!}\Big[\sum_{i=1}^{n_+} \alpha_i^2\Big]^{\frac{k}{2}}.$$

Since $\sum\limits_{i=n_++1}^{n} (\alpha_i)^2 \geqslant 1$. Consequently,

$$EE_h(M) \leqslant n + \sum_{k\geqslant 1} \frac{1}{k!}\Big[2m-1\Big]^{\frac{k}{2}} = n - 1 + e^{\sqrt{2m-1}}.$$

∎

Theorem 3.2. Let M be a mixed graph of order n and size m, then

$$\sqrt{n^2+4m} \leqslant EE_h(M) \leqslant n - 1 + e^{\sqrt{2m}}. \tag{9}$$

Proof: Lower bound Directly from the Hermitian Estrada index, we get

$$EE_h(M)^2 = \sum_{i=1}^{n} e^{2\alpha_i} + 2\sum_{i<j} e^{\alpha_i}e^{\alpha_j}. \tag{10}$$

In view of the inequality between the arithmetic and geometric means,

$$\begin{aligned} 2\sum_{i<j} e^{\alpha_i}e^{\alpha_j} &\geqslant n(n-1)\Big(\prod_{i<j} e^{\alpha_i}e^{\alpha_j}\Big)^{\frac{2}{n(n-1)}} \\ &= n(n-1)\left[\Big(\prod_{i=1}^{n} e^{\alpha_i}\Big)^{n-1}\right]^{\frac{2}{n(n-1)}} \\ &= n(n-1)\Big(e^{\sum_{i=1}^{n}\alpha_i}\Big)^{\frac{2}{n}}, \quad \text{by} \sum_{i=1}^{n}\alpha_i = 0 \\ &= n(n-1). \end{aligned} \tag{11}$$

By means of a power-series expansion, and bearing in mind the properties of N_1 and N_2, we get

$$\sum_{i=1}^{n} e^{2\alpha_i} = \sum_{i=1}^{n}\sum_{k\geqslant 0} \frac{(2\alpha_i)^k}{k!} = n + 4m + \sum_{i=1}^{n}\sum_{k\geqslant 3} \frac{(2\alpha_i)^k}{k!}.$$

Because we are aiming at a (as good as possible) lower bound, it may look plausible to replace $\sum_{k\geqslant 3} \frac{(2\alpha_i)^k}{k!}$ by $8\sum_{k\geqslant 3} \frac{(\alpha_i)^k}{k!}$. However, instead of $8 = 2^3$ we shall use a multiplier $\Gamma \in [0, 8]$, so as to arrive at

$$\sum_{i=1}^{n} e^{2\alpha_i} \geqslant n + 4m + \Gamma \sum_{i=1}^{n}\sum_{k\geqslant 3} \frac{(\alpha_i)^k}{k!}$$
$$= n + 4m - \Gamma n - \Gamma m + \Gamma \sum_{i=1}^{n}\sum_{k\geqslant 0} \frac{(\alpha_i)^k}{k!}$$

in other words,

$$\sum_{i=1}^{n} e^{2\alpha_i} \geqslant (1-\Gamma)n + (4-\Gamma)m + \Gamma EE_h(M). \tag{12}$$

By substituting (11) and (12) back into (10), and solving for $EE_h(M)$ we obtain

$$EE_h(M) \geqslant \frac{\Gamma}{2} + \sqrt{(n-\frac{\Gamma}{2})^2 + (4-\Gamma)m}. \tag{13}$$

It is elementary to show that for $n \geqslant 2$ and $m \geqslant 1$ the function

$$f(x) := \frac{x}{2} + \sqrt{(n-\frac{x}{2})^2 + (4-x)m}$$

monotonically decreases in the interval $[0, 8]$. Consequently, the best lower bound for $EE_h(M)$ is attained not for $\Gamma = 8$. Setting $\Gamma = 0$ into (13) we arrive at the first half of Theorem 3.2.

Upper bound By definition of the Hermitian Estrada index

$$\begin{aligned}
EE_h(M) &= n + \sum_{i=1}^{n}\sum_{k\geqslant 1} \frac{(\alpha_i)^k}{k!} \leqslant n + \sum_{i=1}^{n}\sum_{k\geqslant 1} \frac{(\mid \alpha_i \mid)^k}{k!} \\
&= n + \sum_{k\geqslant 1} \frac{1}{k!} \sum_{i=1}^{n} \Big[(\alpha_i)^2\Big]^{\frac{k}{2}} \leqslant n + \sum_{k\geqslant 1} \frac{1}{k!} \left[\sum_{i=1}^{n} (\alpha_i)^2\right]^{\frac{k}{2}} \\
&= n + \sum_{k\geqslant 1} \frac{1}{k!} \Big(2m\Big)^{\frac{k}{2}} \\
&= n - 1 + \sum_{k\geqslant 0} \frac{\Big(\sqrt{2m}\Big)^k}{k!} \\
&= n - 1 + e^{\sqrt{2m}}.
\end{aligned}$$

Which directly leads to the right-hand side inequality in (9). By this the proof of Theorem 3.2

is completed. ∎

Theorem 3.3. Let M be a mixed graph of order n, size m, and $\alpha_1, \alpha_2, ..., \alpha_n$ be the Hermitian spectrum of $H(M)$, then

$$EE_h(M) \leqslant n - 1 + e^{\sqrt[4]{2m(\alpha_n^2+\alpha_1^2)-n\alpha_1^2\alpha_n^2}}.$$

Proof: By definition of the Hermitian Estrada index, we have

$$\begin{aligned}
EE_h(M) = \sum_{i=1}^{n} e^{\alpha_i} = \sum_{i=1}^{n}\sum_{k=0}^{\infty} \frac{\alpha_i^k}{k!} &\leqslant n + \sum_{i=1}^{n}\sum_{k=1}^{\infty} \frac{\mid \alpha_i \mid^k}{k!} \\
&= n + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i=1}^{n} (\alpha_i^4)^{\frac{k}{4}} \\
&\leqslant n + \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{n} \alpha_i^4 \right)^{\frac{k}{4}} \\
&= n + \sum_{k=1}^{\infty} \frac{1}{k!} (N_4)^{\frac{k}{4}} \\
&= n - 1 + \sum_{k=0}^{\infty} \frac{\sqrt[4]{N_4^k}}{k!} \\
&= n - 1 + e^{\sqrt[4]{N_4}}.
\end{aligned}$$

∎

Therefore, by Lemma 3, we have

$$EE_h(M) \leqslant n - 1 + e^{\sqrt[4]{N_4}} \leqslant n - 1 + e^{\sqrt[4]{2m(\alpha_n^2+\alpha_1^2)-n\alpha_1^2\alpha_n^2}}.$$

Theorem 3.4. Let M be a mixed graph of size m, then

$$EE_h(M) \leqslant e^{\sqrt{2m}}. \tag{14}$$

Proof: By definition of the Hermitian Estrada index, we have

$$EE_h(M) = \sum_{i=1}^{n} e^{\alpha_i} \leqslant \sum_{i=1}^{n} e^{|\alpha_i|} = \sum_{i=1}^{n}\sum_{k\geqslant 0} \frac{(\mid \alpha_i \mid)^k}{k!}$$

$$
\begin{aligned}
&= \sum_{k\geqslant 0} \frac{1}{k!} \sum_{i=1}^{n} (\mid \alpha_i \mid)^k \\
&\leq \sum_{k\geqslant 0} \frac{1}{k!} (\sum_{i=1}^{n} (\mid \alpha_i \mid)^2)^{\frac{k}{2}} \qquad \text{(by Inequality 6)} \\
&= \sum_{k\geqslant 0} \frac{1}{k!} (\sum_{i=1}^{n} (\alpha_i)^2)^{\frac{k}{2}} \\
&= \sum_{k\geqslant 0} \frac{1}{k!} (2m)^{\frac{k}{2}} \qquad \text{(by Equality 2)} \\
&= \sum_{k\geqslant 0} \frac{1}{k!} (\sqrt{2m})^k \\
&= e^{\sqrt{2m}}.
\end{aligned}
$$

∎

4 Bound for the Hermitian Estrada index involving the Hermitian energy

In this section, we investigate the relations between the Hermitian Estrada index and the Hermitian energy.

At first, we state the following useful lemma.

Lemma 4.1. Let M be a mixed graph of order n, size m and $\alpha_1, \alpha_2, ..., \alpha_n$ be the Hermitian spectrum of $H(M)$, then

$$\alpha_1 \geqslant \frac{2m}{E_h(M)}. \tag{15}$$

Proof: Let a_i, b_i be decreasing non-negative sequences with $a_1, b_1 \neq 0$ and w_i a non-negative sequence, for $i = 1, 2, ..., n$. Then the following inequality is valid (see [8] p. 85)

$$\sum_{i=1}^{n} w_i a_i^2 \sum_{i=1}^{n} w_i b_i^2 \leqslant max \left\{ b_1 \sum_{i=1}^{n} w_i a_i, a_1 \sum_{i=1}^{n} w_i b_i \right\} \sum_{i=1}^{n} w_i a_i b_i. \tag{16}$$

For $a_i = b_i := |\alpha_i|$, and $w_i := 1$, $i = 1, 2, ..., n$, inequality (16) becomes

$$\sum_{i=1}^{n} |\alpha_i|^2 \sum_{i=1}^{n} |\alpha_i|^2 \leqslant \max \left\{ \alpha_1 \sum_{i=1}^{n} |\alpha_i|, \alpha_1 \sum_{i=1}^{n} |\alpha_i| \right\} \sum_{i=1}^{n} |\alpha_i|^2.$$

Since

$$\sum_{i=1}^{n} |\alpha_i|^2 = \sum_{i=1}^{n} \alpha_i^2 = 2m,$$

and

$$\sum_{i=1}^{n} |\alpha_i| = E_h(M),$$

from the above inequality directly follows the assertion of Lemma 4.1. ∎

We are now ready to give some new bounds for $EE_h(M)$.

Theorem 4.2. Let M be a mixed graph of order n and size m, then

$$EE_h(M) \geqslant e^{\frac{2m}{E_h(M)}} + \frac{n-1}{e^{\frac{\frac{2m}{E_h(M)}}{n-1}}}.$$

Proof: By definition of the Hermitian Estrada index and using arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
EE_h(M) &= e^{\alpha_1} + e^{\alpha_2} + \cdots + e^{\alpha_n} \\
&\geqslant e^{\alpha_1} + (n-1)\left(\prod_{i=2}^{n} e^{\alpha_i}\right)^{\frac{1}{n-1}} \\
&= e^{\alpha_1} + (n-1)e^{\frac{\sum_{i=2}^{n} e^{\alpha_i}}{n-1}} \\
&= e^{\alpha_1} + (n-1)e^{\frac{-\alpha_1}{n-1}}.
\end{aligned}$$

Now we consider the following function

$$f(x) = e^x + \frac{n-1}{e^{\frac{x}{n-1}}}$$

for $x > 0$. We have

$$f(x) \geqslant e^x + \frac{n-1}{e^{\frac{x}{n-1}}}.$$

It is easy to see that f is an increasing function for $x > 0$. By Lemma 4.1, we obtain

$$EE_h(M) \geqslant e^{\frac{2m}{E_h(M)}} + \frac{n-1}{e^{\frac{\frac{2m}{E_h(M)}}{n-1}}}.$$

∎

Theorem 4.3. The Hermitian Estrada index $EE_h(M)$ and the Hermitian energy $E_h(M)$ satisfy the following inequality:

$$\frac{1}{2}E_h(M)(e-1)+(n-n_+)\leqslant EE_h(M)\leqslant n-1+e^{\frac{E_h(M)}{2}}. \tag{17}$$

Proof: Lower bound Since $e^x \geqslant 1+x$, equality holds if and only if $x=0$ and $e^x \geqslant ex$, equality holds if and only if $x=1$, we have

$$\begin{aligned} EE_h(M) &= \sum_{i=1}^{n} e^{\alpha_i} = \sum_{\alpha_i>0} e^{\alpha_i} + \sum_{\alpha_i\leqslant 0} e^{\alpha_i} \\ &\geqslant \sum_{\alpha_i>0} e\alpha_i + \sum_{\alpha_i\leqslant 0}(1+\alpha_i) \\ &= e(\alpha_1+\alpha_2+\cdots+\alpha_{n_+})+(n-n_+)+(\alpha_{n_++1}+\cdots+\alpha_n) \\ &= (e-1)(\alpha_1+\alpha_2+\cdots+\alpha_{n_+})+(n-n_+)+\sum_{i=1}^{n}\alpha_i \\ &= \frac{1}{2}E_h(M)(e-1)+(n-n_+). \end{aligned}$$

Upper bound By definition of the Hermitian energy index

$$EE_h(M)\leqslant n+\sum_{k\geqslant 1}\frac{1}{k!}\sum_{i=1}^{n_+}(\alpha_i)^k \leqslant n+\sum_{k\geqslant 1}\frac{1}{k!}\left(\sum_{i=1}^{n_+}\alpha_i\right)^k = n-1+e^{\frac{E_h(M)}{2}}.$$

∎

Theorem 4.4. Let G be a graph with the largest eigenvalue α_1 and let p, τ and q be, respectively, the number of positive, zero and negative eigenvalues of G. Then

$$EE_h(M)\geqslant e^{\alpha_1}+\tau+(p-1)e^{\frac{E_h(M)-2\alpha_1}{2(p-1)}}+qe^{-\frac{E_h(M)}{2q}}. \tag{18}$$

Proof: Let $\alpha_1 \geqslant \cdots \geqslant \alpha_p$ be the positive, and $\alpha_{n-q+1}, \ldots, \alpha_n$ be the negative eigenvalues of G. As the sum of eigenvalues of a graph is zero, one has

$$E_h(M) = 2\sum_{i=1}^{p}\alpha_i = -2\sum_{i=n-q+1}^{n}\alpha_i.$$

By the $arithmetic$-$geometric$ mean inequality, we have

$$\sum_{i=2}^{p} e^{\alpha_i} \geqslant (p-1)e^{\frac{(\alpha_2+\cdots+\alpha_p)}{(p-1)}} = (p-1)e^{\frac{E_h(M)-2\alpha_1}{2(p-1)}}. \tag{19}$$

Similarly,

$$\sum_{i=n-q+1}^{n} e^{\alpha_i} \geqslant qe^{-\frac{E_h(M)}{2q}}. \tag{20}$$

For the zero eigenvalues, we also have

$$\sum_{i=p+1}^{n-q} e^{\alpha_i} = \tau.$$

So we obtain

$$EE_h(M) \geqslant e^{\alpha_1} + \tau + (p-1)e^{\frac{E_h(M)-2\alpha_1}{2(p-1)}} + qe^{-\frac{E_h(M)}{2q}}.$$

∎

Theorem 4.5. Let M be a mixed graph of order n and size m, then

$$EE_h(M) - E_h(M) \leqslant n - 1 - \sqrt{2m} + e^{\sqrt{2m}}.$$

Proof: By definition of the Hermitian Estrada index, we have

$$EE_h(M) = n + \sum_{i=1}^{n}\sum_{k\geqslant 1} \frac{(\alpha_i)^k}{k!} \leqslant n + \sum_{i=1}^{n}\sum_{k\geqslant 1} \frac{|\ \alpha_i\ |^k}{k!}.$$

Moreover, by considering the Hermitian energy, we get

$$EE_h(M) \leqslant n + E_h(M) + \sum_{i=1}^{n}\sum_{k\geqslant 2} \frac{|\ \alpha_i\ |^k}{k!}.$$

Hence

$$\begin{aligned} EE_h(M) - E_h(M) &\leqslant n + \sum_{i=1}^{n}\sum_{k\geqslant 2} \frac{|\ \alpha_i\ |^k}{k!} \\ &\leqslant n - 1 - \sqrt{2m} + e^{\sqrt{2m}}. \end{aligned}$$

∎

Theorem 4.6. Let M be a mixed graph of order n. Then

$$EE_h(M) \leqslant n - 1 + e^{E_h(M)}.$$

Proof: By definition of the Hermitian Estrada index, we have

$$\begin{aligned} EE_h(M) &\leqslant n + \sum_{i=1}^{n} \sum_{k\geqslant 1} \frac{|\ \alpha_i\ |^k}{k!} \\ &\leqslant n + \sum_{k\geqslant 1} \frac{1}{k!} \left(\sum_{i=1}^{n} |\ \alpha_i\ |^k \right) \\ &= n + \sum_{k\geqslant 1} \frac{(E_h(M))^k}{k!} \end{aligned}$$

which implies

$$EE_h(M) \leqslant n - 1 + e^{E_h(M)}.$$

∎

Concluding Remarks

In this paper, the Hermitian Estrada index of a mixed graph is introduced. Also the Hermitian energy and the Hermitian Estrada index are studied and we obtained some bounds for the Hermitian Estrada index of mixed graphs. Finally, we investigated the relations between the Hermitian Estrada index and the Hermitian energy of mixed graphs.

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