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On The Sum-eccentricity Energy of a Graph

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Abstract

In this paper, we obtain the coefficient c_2 in the characteristic polynomial of some well-known graphs and discuss the relation between c_2 and the sum-eccentricity energy of a graph. We introduce the concept of hyper sum-eccentricity energetic graph. A new upper bound for the sum-eccentricity energy $ES_e(G)$ is obtained. The sum-eccentricity energy of some well-known graphs are derived. We show that $ES_e(S_{1,r}) = 6r^{\frac{1}{2}}$ for the star $S_{1,r}$.

Key words: Distance in graphs, Sum-eccentricity matrix, Sum-eccentricity eigenvalues, Sum-eccentricity energy of a graph, Hyper Sum-eccentricity energitec graph.

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1 Introduction

In this paper, all the graphs are assumed to be simple connected graphs. Let G be a simple graph with n vertices, m edges, and let $A = (a_{ij})$ be its adjacency matrix, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A are the (ordinary) eigenvalues of the graph G [8]. Since A is a symmetric matrix with zero trace, these eigenvalues are real with sum equal to zero. The energy of the graph G is defined in [8], as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

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Details on the theory of graph energy can be found in the reviews [2, 3, 5], whereas details on its chemical applications in the book [7], and in the review [4]. The energy of the complete graph K_n is equal to $2(n-1)$. An n-vertex graph G is said to be hyperenergetic if $E(G) > E(K_n)$ [6]. Details on hyperenergetic graphs can be found in the review [2].

The sum-eccentricity matrix of a graph G is denoted by $S_e(G)$ and defined as $S_e(G) = (s_{ij})$ [10], where

$$s_{ij} = \begin{cases} e(v_i) + e(v_j), & if\ v_iv_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mu_1, \mu_2, \ldots, \mu_n$, are the sum-eccentricity eigenvalues, then the sum-eccentricity energy is

$$ES_e(G) = \sum_{i=1}^{n} |\mu_i|.$$

The distance $d(u, v)$ between any two vertices u and v in a graph G is the length of the shortest path connecting them. The eccentricity of a vertex $v \in G$ is $e(v) = max\{d(u, v) : u \in V(G)\}$. The radius of G is $r(G) = min\{e(v) : v \in V(G)\}$ and the diameter of G is $D(G) = max\{e(v) : v \in V(G)\}$. Hence $r(G) \le e(v) \le D(G)$, for every $v \in V(G)$. A vertex v in a connected graph G is central vertex if $e(v) = r(G)$, while a vertex v in a connected graph G is peripheral vertex if $e(v) = D(G)$, a graph G is said to be self centered graph if $e(v) = r(G) = D(G)$ [10].

Lemma 1.1. [10] Let G be a graph of order n and let

$$P(G, \mu) = c_0\mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + ... + c_n$$

be the characteristic polynomial of the sum-eccentricity matrix of G. Then

$$c_2 = - \sum_{i=1,\ i<j}^{n} (e(v_i) + e(v_j))^2,\ for\ all\ v_iv_j \in E.$$

2 Elementary Results

Theorem 2.1. Let G be a graph of order n and size m. Then

$$r^2(G) \le \frac{-c_2}{4m} \le D^2(G),$$

with equality $r^2(G) = \frac{-c_2}{4m} = D^2(G)$, holds if and only if G is a self centered graph.

Proof: We have $r(G) \leq e(v_i) \leq D(G)$, for all $i = 1, 2, \ldots, n$, so

$$(2r(G))^2 \leq (e(v_i) + e(v_j))^2 \leq (2D(G))^2, \ i, j = 1, 2, ..., n.$$

Since $r(G) \geq 1$ and $v_iv_j \in E$, using lemma 1.1, we get

$$4m(r(G))^2 \leq \sum_{i=1, i<j}^{n} (e(v_i) + e(v_j))^2 \leq 4m(D(G))^2$$

hence

$$r^2(G) \leq \frac{-c_2}{4m} \leq D^2(G).$$

For equality to hold, let $r^2(G) = \frac{-c_2}{4m} = D^2(G)$, then $r(G) = D(G) = e(v_i)$ for all $i = 1,\ 2,\ ...,\ n$, hence G is a self centered graph. On the other hand, let G be a self centered graph, then $r(G) = D(G)$, which implies easily $r^2(G) = \frac{-c_2}{4m} = D^2(G)$. ■

We investigate the values of the sum-eccentricity energy of some well-known graphs.

Theorem 2.2. Let G be a complete bipartite graph, $G = K_{a,b}$. Then $c_2 = -4^2ab$, where a and b are integers with $a \ and\ b \geq 2$.

Proof: Lemma 1.1, gives

$$c_2 = -\sum_{i=1,\ i<j}^{n} (e(v_i) + e(v_j))^2,$$

where $v_iv_j \in E$, and in $K_{a,b}$, each $e(v_i) = 2$, $i = 1, 2, \ldots, a + b$.
Hence $e(v_i) + e(v_j) = 4$, $i, j = 1, 2, \ldots, a + b$, $i \neq j$. It is stay to compute the number of elements in the summation. If we name the first set of vertices by A and the second set by B, since each vertex in the set A adjacent to each vertex in the set B, then $m = ab$. Thus

$$c_2 = -\sum_{i=1}^{ab} 4^2 = -4^2ab.$$

■

Corollary 2.3. For the star $S_{1,r}$, $r \geq 2$,

$$c_2 = -9r.$$

Proof: Using the same equation in lemma 1.1, and the fact that $e(v_i) + e(v_j) = 3$, in $S_{1,r}$ and since each v_i adjacent only to the point in the center, then $m = r$, thus

$$c_2 = \sum_{i=1}^{r} 3^2 = -9r.$$

■

Theorem 2.4. Let $G = K_{a,b}$, then the eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$, will be

$$S_eSp(K_{a,b}) = \begin{bmatrix} 4\sqrt{ab} & 0 & -4\sqrt{ab} \\ 1 & n-2 & 1 \end{bmatrix}$$

Proof: Using lemma 1.1, we get $c_k = 0$, $k = 1, 3, 4, \cdots, n$.
Hence the sum-eccentricity characteristic polynomial is

$$\begin{aligned} \phi(K_{a,b}, \mu) &= c_0\mu^n + c_1\mu^{n-1} + \cdots + c_n \\ &= \mu^n - 4^2 ab\mu^{n-2}. \end{aligned}$$

Which implies that

$$S_eSp(K_{a,b}) = \begin{bmatrix} 4\sqrt{ab} & 0 & -4\sqrt{ab} \\ 1 & n-2 & 1 \end{bmatrix}$$

■

Corollary 2.5. For the complete bipartite graph $K_{a,b}$, the sum-eccentricity energy is $ES_e(K_{a,b}) = 8\sqrt{ab}$.

We study the complete bipartite graph $K_{a,b}$, in case of a and b are both grater than 1. The following result gives exact value in case of $a = 1$ of the sum-eccentricity energy of the star $S_{1,r}$.

Theorem 2.6. The sum-eccentricity energy of the star $S_{1,r}$ is

$$ES_e(S_{1,r}) = 6r^{\frac{1}{2}}.$$

Proof: We claim that the characteristic polynomial of the star $S_{1,r}$ is

$$\phi(S_{1,r}, \mu) = \mu^{r+1} - 9r\mu^{r-1}.$$

We will show this by the method of mathematical induction,
if we assume $r = 1$, then

$$\begin{aligned}\phi(S_{1,r}, \mu) &= \begin{vmatrix} \mu & -3 \\ -3 & \mu \end{vmatrix} \\ &= \mu^2 - 9.\end{aligned}$$

Now we assume that it is true for $r = k$, i.e. $\phi(S_{1,k}, \mu) = \mu^{k+1} - 9k\mu^{k-1}$,
so for $r = k + 1$ we have

$$\phi(S_{1,k+1}, \mu) = \begin{vmatrix} \mu & -3 & -3 & \cdots & -3 \\ -3 & \mu & 0 & \cdots & 0 \\ -3 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -3 & 0 & 0 & \cdots & \mu \end{vmatrix}_{k+2\times k+2}$$

now exchanging the first row and the second row and doing the same for the first column and the second column we get

$$\begin{aligned}\phi(S_{1,k+1}, \mu) &= \begin{vmatrix} \mu & -3 & 0 & \cdots & 0 \\ -3 & \mu & -3 & \cdots & -3 \\ 0 & -3 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -3 & 0 & \cdots & \mu \end{vmatrix}_{k+2\times k+2} \\ &= \mu(\mu^{k+1} - 9k\mu^{k-1}) + 3\begin{vmatrix} -3 & -3 & -3 & \cdots & -3 \\ 0 & \mu & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{vmatrix}_{k+1\times k+1} \\ &= \mu(\mu^{k+1} - 9k\mu^{k-1}) + 3(-3\mu^k) \\ &= \mu^{k+2} - 9k\mu^k - 9\mu^k \\ &= \mu^{k+2} - 9\mu^k(k + 1).\end{aligned}$$

Hence

$$\phi(S_{1,r},\mu) = \mu^{r+1} - 9r\mu^{r-1},$$

so the eigenvalues of $S_{1,r}$ are

$$S_eSp(S_{1,r}) = \begin{bmatrix} 3\sqrt{r} & 0 & -3\sqrt{r} \\ 1 & r-1 & 1 \end{bmatrix}$$

which gives

$$ES_e(S_{1,r}) = 6r^{\frac{1}{2}}.$$

■

3 A New Bound for the Sum-eccentricity Energy of a Graph

In this section we use Gershgorin disc theorem and find a new upper bound for the sum-eccentricity energy of a graph G.

Definition 3.1. [11](Gershgorin disc) Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, and let D_i, be the closed disc in the real plane centered at a_{ii} with radius $r_i = \sum_{i=1,i\neq j}^{n} |a_{ij}|$, then D_i is called Gershgorin disc.

Theorem 3.2. [11] If $\lambda_1, \lambda_2, \ldots, \lambda_n$, are the eigenvalues of $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, then each Gershgorin disc must have at least one eigenvalue i.e. $D_i = \{x \in \mathbb{R} : |x - a_{ii}| \leq r_i\} \equiv D(a_{ii}, r = r_i)$.

Theorem 3.3. Let G be a graph of order n and size m then

$$ES_e(G) \leq 2 \sum_{i=1,i<j}^{n} (e(v_i) + e(v_j)),$$

where $v_iv_j \in E$.

Proof: For the graph G we have $S_e(G)$ is a real symmetric matrix with $s_{ii} = 0, i = 1, 2, \ldots, n$, so the Gershgorin disc is $D_i = \{x \in \mathbb{R} : |x| \leq r_i\}$. Using Theorem3.2, if we arrange $r_1, r_2, \ldots, r_n$, and $|\mu_1|, |\mu_2|, \ldots, |\mu_n|$, in a non-increasing manner, we get $|\mu_1| \leq r_1, |\mu_2| \leq r_2, \ldots, |\mu_n| \leq r_n$.
Hence

$$\sum_{i=1}^{n} |\mu_i| \leq \sum_{i=1}^{n} r_i$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} s_{ij}$$
$$= 2\sum_{i=1, i<j}^{n} s_{ij}$$
$$= 2\sum_{i=1, i<j}^{n} (e(v_i)+e(v_j)),\ for\ all\ v_iv_j \in E.$$

Thus

$$ES_e(G) \leq 2\sum_{i=1, i<j}^{n} (e(v_i)+e(v_j)),$$

where $v_iv_j \in E$. ∎

4 Hyper Sum-eccentricity Energetic Graphs

The energy of the complete graph K_n is $2(n-1)$, [8].

Definition 4.1. [2] A graph G on n vertices is said to be hyperenergetic if

$$E(G) > 2(n-1).$$

The sum-eccentricity energy of the complete graph K_n is $ES_e(K_n) = 4(n-1)$, [10].

Definition 4.2. A graph G on n vertices is said to be hyper sum-eccentricity energetic graph if

$$ES_e(G) > 4(n-1).$$

Claim 4.3. There exist a complete bipartite graph that is hyper sum-eccentricity energetic graph.

Theorem 4.4. The complete bipartite graph $K_{a,a}$, a is a positive integer $a \geq 1$, is a hyper sum-eccentricity energetic graph.

Proof: We have from Corollary2.5, $ES_e(K_{a,a}) = 8a$, so the total number of vertices in $K_{a,a}$ is $n = 2a$. Hence

$$ES_e(K_{a,a}) = 8a$$

$$> 4(2a-1)$$
$$= ES_e(K_n).$$

So $K_{a,a}$ is a hyper sum-eccentricity energetic graph. ∎

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