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# On the Downhill Domination Polynomial of Graphs

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**Abstract**

Graph polynomials have been developed for calculate the structural information of networks using combinatorial graph parameters and to characterize the graphs. Many problems in graph theory and discrete mathematics can be treated and solved in a rather efficient manner by making use of polynomials. In this research work, we introduce and study a new graph polynomial called downhill domination polynomial of a graph along with this new polynomial, we define the downhill domination roots of a graph. The downhill domination polynomial with its roots of some standard families of graphs and few graphs of cycle related graphs such as complete graph, tadpole graph, lollipop graph, gear graph and barbell graph are obtained. Downhill domination polynomials for book graph and stacked book graph are established. Finally, we get general result for determine the downhill domination polynomial of any graph of $n$ vertices with $\gamma_{dn}(G) = 1$ and has $r$ minimum downhill dominating sets, also for a graph of $n$ vertices with unique minimum downhill dominating set with size $s$ along with these general results, graphical representation are presented to show the behavior and positions of the downhill domination roots.

**Key words:** Downhill domination polynomial, Downhill domination roots

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## 1 Introduction

In this paper, we concerned only about connected simple graphs $G = (V, E)$ which are finite, undirected with no loops and multiple edges. The degree of a vertex $v$ in a graph is the number of edges incident with it, denoted by $deg(v)$. A $u - v$ path $P$ in a graph $G$ is a sequence of vertices in $G$, starting with $u$ and ending at $v$, such that consecutive vertices in $P$ are adjacent, and no vertex is repeated. We say that a path $\pi = v_1, v_2, ...v_{k+1}$ is a downhill path if for every i, $1 \leq i \leq k$, $deg(v_i) \geq deg(v_{i+1})$ [4].
The following notations and different types of graphs well known in the literature [3] and [5]. A complete multipartite graph is a multipartite graph such that any two vertices that are not in the same part have an edge connecting them. We will denote a complete multipartite graph with $k$ parts by $K_{n_1,n_2,...,n_k}$ where $n_i$ is the number of vertices in the $i^{th}$ part of the graph. A Double star is the graph obtained from $K_2$ by joining $s$ pendent edges to one end and $r$ pendent edges to the other end of $K_2$. A wheel $W_{n+1}$, $n \geqslant 3$ is the join of $C_n$ and $K_1$. A helm graph, denoted by $H_n$, is a graph obtained from $W_{n+1}$ by attaching an end edge to each rim vertex of $W_{n+1}$, where the vertices corresponding to $C_n$ are known as rim vertices. The gear graph is a wheel graph with a vertex added between each pair adjacent graph vertices of the outer cycle. The gear graph $G_n$ has $2n+1$ vertices and $3n$ edges. The sierpinski sieve graph $S_n$ is the graph obtained from the connectivity of the sierpinski sieve. The graph has $\frac{3(3^{n-1}+1)}{2}$ vertices and $3^n$ edges. The tadpole graph $T_{r,s}$ is obtained by joining a cycle $C_r$ and a path $P_s$ by a bridge, where $r \geqslant 3$ and $s \geqslant 1$. The lollipop graph $L_{r,s}$ is obtained by joining a complete graph $K_r$ and a path $P_s$ by a bridge, where $r \geqslant 3$ and $s \geqslant 1$. The $n$-barbell graph $B_n$ is obtained by connecting two copies of a complete graph $K_n$ by a bridge. The Cartesian product $G$ of two graphs $G_1$ and $G_2$, denoted $G_1 \Box G_2$, has vertex set $V(G) = V(G_1) \Box V(G_2)$, and two distinct vertices $(a, b)$ and $(c, d)$ of $G_1 \Box G_2$ are adjacent if either $a = c$ and $bd \in E(G_2)$, or $b = d$ and $ac \in (G_1)$.

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is a dominating set if every vertex $v \in V$ is an element of $S$ or adjacent to an element of $S$. A downhill dominating set, abbreviated DDS, is a set $S \subseteq V$ having the property that every vertex $v \in V$ lies on a downhill path originating from some vertex in $S$. The downhill domination number $\gamma_{dn}(G)$ equals the minimum cardinality of a DDS of $G$ [4].

Graph polynomials have been developed for measuring structural information of networks using combinatorial graph invariants and for characterizing graphs. Various problems in graph

theory and discrete mathematics can be treated and solved in a rather efficient manner by making use of polynomials.
Various graph polynomials have been proven useful in discrete mathematics, engineering, information sciences, mathematical chemistry, and related disciplines. Numerous graph polynomials were introduced in the literature, several of them also turned out to be applicable in mathematical chemistry. One of the interested graph polynomials is the domination polynomial of graphs [1].

The downhill domination concept and the huge application of graph polynomial motivated us to introduce a new graph polynomial called downhill domination polynomial of graph.
In this research work, we introduce new graph polynomial called downhill domination polynomial and along with this polynomial, we define the downhill domination roots of a graph exact values and expressions for the standard families of graphs, and some graph operations are obtained.

## 2 Downhill Domination Polynomial

In this section, we define the downhill domination polynomial of graph and obtain this polynomial along with its roots for some families of standard graphs.

**Definition 2.1.** For any graph $G$ of $n$ vertices, the downhill domination polynomial of $G$ is define by $DW(G,x) = \sum_{i=\gamma_{dn}(G)}^{n} dw(G,i)x^i$, where $dw(G,i)$ is the number of downhill dominating sets of size $i$ in $G$. The set of roots of $DW(G,x)$ is called downhill domination roots of graph $G$ and denoted by $Z_{dw}(G)$.

**Observation 2.2.** Let $G$ be a connected regular graph of $n$ vertices. Then, $DW(G,x) = (1+x)^n - 1$.

**Corollary 2.3.**

i. Let $G$ be any cycle graph $C_n$, where $n \geq 3$. Then, $DW(G,x) = (1+x)^n - 1$.

ii. Let $G$ be any complete graph $K_n$. Then, $DW(G,x) = (1+x)^n - 1$.

**Observation 2.4.** Let $G \cong P_n$ be a path of order $n$, where $n \geqslant 4$. Then, $DW(G,x) = (1+x)^2((1+x)^{n-2} - 1)$.

**Proposition 2.5.** Let $G \cong K_{r,s}$ be a complete bipartite of $r+s$ vertices, where $s, r \geq 2$. Then,

$$DW(G,x) = \begin{cases} (1+x)^{s+r} - 1 & if\, r = s; \\ x^r(1+x)^s & if\, r < s. \end{cases}$$

**Proof:** Let $G \cong K_{r,s}$ be a complete bipartite of $r+s$ vertices, where $s, r \geq 2$. We have two cases:

**Case 1.** If $s = r$, then $G$ is connected regular graph. Hence, $DW(G,x) = (1+x)^{r+s} - 1$.

**Case 2.** If $r < s$, then $\gamma_{dn}(G) = r$. In this case there is only one minimum downhill dominating set of size $r$. Then, for $i = r+1$ there are $\binom{s}{1}$ downhill dominating sets of size $r+1$. Also, for $i = r+2$, there are $\binom{s}{2}$ downhill dominating sets of size $r+2$. This means, for $i = r+1, r+2, ..., r+s$, there are $\binom{s}{i-r}$ downhill dominating sets of size $i$. Thus, we have

$$\begin{aligned} DW(G,x) &= \binom{s}{0}x^r + \binom{s}{1}x^{r+1} + ... + \binom{s}{s}x^{r+s} \\ &= x^r\left[\binom{s}{0}x^0 + \binom{s}{1}x + ... + \binom{s}{s}x^s\right] \\ &= x^r(1+x)^s. \end{aligned}$$

∎

**Theorem 2.6.** Let $G \cong K_{n_1,n_2,...,n_k}$ be complete multipartite graph of $n$ vertices, where $n = \sum_{i=1}^{k} n_i$. Then,

$$DW(G,x) = \begin{cases} (1+x)^n - 1 & if \quad n_1 = n_2 = ... = n_k; \\ x^{n_1}(1+x)^{n-n_1} & if \quad n_1 < n_2 < ... < n_k; \\ (1+x)^{n-d}((1+x)^d - 1) & if \quad n_1 = n_2 = ... = n_i < ... < n_k\text{ , where } d = \sum_{j=1}^{i} n_j. \end{cases}$$

**Proof:** Let $G \cong K_{n_1,n_2,...,n_k}$ be complete multipartite graph of $n$ vertices, where $n = \sum_{i=1}^{k} n_i$. We have three cases:

**Case 1.** $n_1 = n_2 = ... = n_k$, then the graph $G$ is connected regular graph. Hence, $DW(G,x) = (1+x)^n - 1$.

**Case 2.** $n_1 < n_2 < ... < n_k$. In this case there is only one minimum downhill dominating sets of size $n_1$. This means, $dw(G,n_1) = 1$. Then, for $i = n_1+1, n_1+2, ..., n$, there are $\binom{n-n_1}{i-n_{n_1}}$ downhill dominating sets of size $i$. Thus, we get

$$DW(G,x) = \binom{n-n_1}{0}x^{n_1} + \binom{n-n_1}{1}x^{n_1+1} + ... + \binom{n-n_1}{n-n_1}x^n$$

$$\begin{aligned} &= x^{n_1}\left[\binom{n-n_1}{0}x^0+\binom{n-n_1}{1}x+\ldots+\binom{n-n_1}{n-n_1}x^{n-n_1}\right] \\ &= x^{n_1}(1+x)^{n-n_1}. \end{aligned}$$

**Case 3.** $n_1 = n_2 = ... = n_i < ... < n_k$ and $d = \sum_{j=1}^{i} n_j$. In this case, there are $d$ minimum downhill dominating sets of size one. Then, we have

$$dw(G,1) = \binom{n}{1} - \binom{n-d}{1} = d.$$

For $i = 2, 3, ..., n-d$. Every downhill dominating set of size $i$ must be contains at least one minimum downhill dominating set. Then,

$$dw(G,2) = \binom{n}{2} - \binom{n-d}{2}, ..., dw(G,n-d) = \binom{n}{n-d} - \binom{n-d}{n-d}.$$

For $i = n-d+1, ..., n$. Any $i$ vertices are downhill dominating set of size $i$, then

$$dw(G,n-d+1) = \binom{n}{n-d+1}, ..., dw(G,n) = \binom{n}{n}.$$

Thus, we get

$$\begin{aligned} DW(G,x) &= \left[\binom{n}{1}-\binom{n-d}{1}\right]x+\ldots+\left[\binom{n}{n-d}-\binom{n-d}{n-d}\right]x^{n-d} \\ &+ \binom{n}{n-d+1}x^{n-d+1}+\ldots+\binom{n}{n}x^n \\ &= \binom{n}{1}x+\ldots+\binom{n}{n-d}x^{n-d}+\ldots+\binom{n}{n}x^n-\left[\binom{n-d}{1}x+\ldots+\binom{n-d}{n-d}x^{n-d}\right] \\ &= (1+x)^n-(1+x)^{n-d} \\ &= (1+x)^{n-d}((1+x)^d-1). \end{aligned}$$

∎

**Proposition 2.7.** Let $G \cong S_n$ be a star of $n+1$ vertices, where $n \geq 2$. Then, $DW(G,x) = x(1+x)^n$. Furthermore, $Z_{dw} = \{0, -1\}$ .

**Proof:** Let $G \cong S_n$ be a star of $n+1$ vertices, where $n \geq 2$. It has only one downhill dominating set of size one. Then, for $i = 2, 3, ..., n+1$, there are $\binom{n}{i-1}$ downhill dominating sets of size $i$. Thus, we have

$$\begin{aligned} DW(G,x) &= \binom{n}{0}x + +\binom{n}{1}x^2 + ... + \binom{n}{n}x^{n+1} \\ &= x\left[\binom{n}{0}x^0 + \binom{n}{1}x + ... + \binom{n}{n}x^n\right] \\ &= x(1+x)^n. \end{aligned}$$

■

**Proposition 2.8.** Let $G \cong S_{m,n}$ be a double star graph of $m+n+2$ vertices, where $m,n \geq 2$. Then,

$$DW(G,x) = \begin{cases} (1+x)^{m+n}(x^2+2x) & if\, n = m; \\ x(1+x)^{d-1} & if\, m > n \end{cases}, \text{where } d = m+n+2.$$

**Proof:** Let $G \cong S_{m,n}$ be a double star of $m+n+2$ vertices, where $m,n \geq 2$. Let $d = m+n+2$, we have two cases:

**Case 1.** If $m = n$. In this case, there are two minimal downhill dominating sets of size one. Then, we have

$$dw(G,1) = \binom{d}{1} - \binom{d-2}{1} = 2.$$

For $i = 2,3,...,d-2$. Every downhill dominating set of size $i$ must be contains at least one minimum downhill dominating set. Then,

$$dw(G,2) = \binom{d}{2} - \binom{d-2}{2}, ..., dw(G,d-2) = \binom{d}{d-2} - \binom{d-2}{d-2}.$$

For $i = d-1, d$. Any $i$ vertices are downhill dominating set of size $i$. Then

$$dw(G,d-1) = \binom{d}{d-1}, dw(G,d) = \binom{d}{d}.$$

Thus, we get

$$\begin{aligned} DW(G,x) &= \left[\binom{d}{1} - \binom{d-2}{1}\right]x + ... + \left[\binom{d}{d-2} - \binom{d-2}{d-2}\right]x^{d-2} + \binom{d}{d-1}x^{d-1} + \binom{d}{d}x^d \\ &= \binom{d}{1}x + .. + \binom{d}{d}x^d - \left[\binom{d-2}{1}x + ... + \binom{d-2}{d-2}x^{d-2}\right] \\ &= (1+x)^d - (1+x)^{d-2} \end{aligned}$$

$$= (1+x)^{d-2}((1+x)^2-1).$$

Hence,

$$DW(G,x) = (1+x)^{m+n}(x^2+2x).$$

**Case 2.** If $m > n$. In this case there is only one downhill dominating set of size one. Then, for $i = 2, 3, ..., d+1$, there are $\binom{d-1}{i-1}$ downhill dominating set of size $i$. Thus, we get

$$\begin{aligned} DW(G,x) &= \binom{d-1}{0}x + \binom{d-1}{1}x^2 + ... + \binom{d-1}{d-1}x^d \\ &= x\left[\binom{d-1}{0}x^0 + \binom{d-1}{1}x + ... + \binom{d-1}{d-1}x^{d-1}\right] \\ &= x(1+x)^{d-1}. \end{aligned}$$

Hence, $DW(G,x) = x(1+x)^{m+n+1}$. ∎

**Proposition 2.9.** Let $G \cong H_n$ be a helm graph of $2n+1$ vertices, where $n \geqslant 5$. Then,$DW(G,x) = x(1+x)^{2n}$. Furthermore, $Z_{dw} = \{0, -1\}$.

**Proof:** Let $G \cong H_n$ be a helm graph of $2n+1$ vertices, where $n \geqslant 5$. It has only one minimum downhill dominating set of size one. Then, for $i = 2, 3, ..., 2n+1$, there are $\binom{2n}{i-1}$ downhill dominating set of size $i$. Thus, we have

$$\begin{aligned} DW(G,x) &= \binom{2n}{0}x + \binom{2n}{1}x^2 + ... + \binom{2n}{2n}x^{2n+1} \\ &= x\left[\binom{2n}{0}x^0 + \binom{2n}{1}x + ... + \binom{2n}{2n}x^{2n}\right] \\ &= x(1+x)^{2n}. \end{aligned}$$

∎

Similarly to Proposition 2.9, we can prove the result for gear graph.

**Proposition 2.10.** Let $G \cong G_n$ be gear graph of $2n+1$ vertices, where $n \geqslant 4$. Then, $DW(G,x) = x(1+x)^{2n}$. Furthermore, $Z_{dw} = \{0, -1\}$.

**Proposition 2.11.** Let $G$ be a Sierpinski Sieve graph of $m$ vertices , where $m = \frac{3(3^{n-1}+1)}{2}$. Then,

$$DW(G,x) = (1+x)^3((1+x)^{m-3}-1).$$

**Proof:** Let $G \cong S_n$ be a Sierpinski Sieve graph of $m$ vertices , where $m = \frac{3(3^{n-1}+1)}{2}$. from the definition of Sierpinski Sieve graph, clearly, there are $m-3$ minimum downhill dominating sets of size one, then, we have

$$dw(G,1) = \binom{m}{1} - \binom{3}{1}.$$

For $i = 2, 3$,

$$dw(G,2) = \binom{m}{2} - \binom{3}{2}, dw(G,3) = \binom{m}{3} - \binom{3}{3}.$$

Now, for $i = 4, 5, ..., m$. Any $i$ vertices are downhill dominating set of size $i$. Then

$$dw(G,4) = \binom{m}{4}, ..., dw(G,m) = \binom{m}{m}.$$

Thus, we get

$$\begin{aligned} DW(G,x) &= \left[\binom{m}{1} - \binom{3}{1}\right] x + \left[\binom{m}{2} - \binom{3}{2}\right] x^2 \\ &+ \left[\binom{m}{3} - \binom{3}{3}\right] x^3 + \binom{m}{4} x^4 + \cdots + \binom{m}{m} x^m \\ &= \binom{m}{1} x + ... + \binom{m}{m} x^m - \left[\binom{3}{1} x + \binom{3}{2} x^2 + \binom{3}{3} x^3\right] \\ &= (1+x)^m - (1+x)^3 \\ &= (1+x)^3((1+x)^{m-3} - 1). \end{aligned} \tag{1}$$

■

**Theorem 2.12.** Let $G \cong T_{m,n}$ be a tadpole graph of $m+n$ vertices. Then, $DW(G,x) = x(1+x)^{m+n-1}$. Furthermore, $Z_{dw}(G) = \{0, -1\}$.

**Proof:** Let $G \cong T_{m,n}$ be a tadpole graph of $m+n$ vertices. It has only one minimum downhill dominating set of size one. Let $d = m+n$, then for $i = 2, 3, ..., d$, there are $\binom{d-1}{i-1}$ downhill dominating set of size $i$. Thus, we get

$$\begin{aligned} DW(G,x) &= \binom{d-1}{0} x + \binom{d-1}{1} x^2 + ... + \binom{d-1}{d-1} x^d \\ &= x \left[\binom{d-1}{0} x^0 + ... + \binom{d-1}{d-1} x^{d-1}\right] \\ &= x(1+x)^{d-1}. \end{aligned}$$

Hence, $DW(G,x) = x(1+x)^{m+n-1}$. ■

**Proposition 2.13.** Let $G \cong L_{m,n}$ be lollipop graph of $m+n$ vertices, then $DW(G,x) =$

$x(1+x)^{m+n-1}$. Furthermore, $Z_{dw}(G) = \{0, -1\}$.

**Proof:** The proof similarly to the the proof Theorem 2.12. ■

**Proposition 2.14.** Let $G$ be barbell graph of $2n$ vertices. Then, $DW(G, x) = (1+x)^{2(n-1)}(x^2+2x)$. Furthermore, $Z_{dw}(G) = \{0, -1, -2\}$.

**Proof:** Let $G$ be barbell graph of $2n$ vertices. There are two vertices of degree $n$ and all the other vertices are of degree $n-1$ so the graph has two minimum downhill dominating sets of size one. Then, we have

$$dw(G, 1) = \binom{2n}{1} - \binom{2n-2}{1}.$$

For $i = 2, 3, ..., 2n-2$. Every downhill dominating set of size $i$ must be at least one of the two vertices of degree $n-1$.

$$dw(G, 2) = \binom{2n}{2} - \binom{2n-2}{2}, ..., dw(G, 2n-2) = \binom{2n}{2n-2} - \binom{2n-2}{2n-2}.$$

For $i = 2n-1, 2n$. Any $i$ vertices are downhill dominating set of size $i$. Then $dw(G, 2n-1) = \binom{2n}{2n-1}$ and $dw(G, 2n) = \binom{2n}{2n}$. Thus, we get

$$\begin{aligned} DW(G, x) &= \left[\binom{2n}{1} - \binom{2n-2}{1}\right] x + ... + \left[\binom{2n}{2n-2} - \binom{2n-2}{2n-2}\right] x^{2n-2} \\ &+ \binom{2n}{2n-1} x^{2n-1} + \binom{2n}{2n} x^{2n} \\ &= \binom{2n}{1} x + ... + \binom{2n}{2n} x^{2n} - \left[\binom{2n-2}{1} x + ... + \binom{2n-2}{2n-2} x^{2n-2}\right] \\ &= (1+x)^{2n} - (1+x)^{2n-2} \\ &= (1+x)^{2n-2}((1+x)^2 - 1). \end{aligned}$$

Hence,

$$DW(G, x) = (1+x)^{2(n-1)}(x^2 + 2x).$$

Also it is obviously that $Z_{dw}(G) = \{0, -1, -2\}$. ■

**Proposition 2.15.** Let $G \cong P_n \square P_m$ be the grid graph of $mn$ vertices, where $m, n \geq 3$. Then,

$$DW(G, x) = (1+x)^{2(n+m-2)}((1+x)^{(m-2)(n-2)} - 1).$$

**Proof:** Let $G \cong P_n \square P_m$ be the grid graph of $mn$ vertices, where $m, n \geq 3$. There are $(m-2)(n-2)$ minimum downhill dominating sets of size one. Let $d = mn$ and $t = (m-2)(n-2)$.

Then, we have
$$dw(G,1) = \binom{d}{1} - \binom{d-t}{1} = t.$$
For $i = 2, 3, ..., d-t$. Every downhill dominating set of size $i$ must contains at least one vertex from the $t$ vertices which is of degree $4$.
$$dw(G,2) = \binom{d}{2} - \binom{d-t}{2}, ..., dw(G, d-t) = \binom{d}{d-t} - \binom{d-t}{d-t}.$$
Now, for $i = d-t+1, ..., d$. Any $i$ vertices are downhill dominating set of size $i$. Then
$$dw(G, d-t+1) = \binom{d}{d-t+1}, ..., dw(G,d) = \binom{d}{d}.$$
Thus, we get
$$\begin{aligned} DW(G,x) &= \left[\binom{d}{1} - \binom{d-t}{1}\right] x + ... + \left[\binom{d}{d-t} - \binom{d-t}{d-t}\right] x^{d-t} \\ &+ \binom{d}{d-t+1} x^{d-t+1} + ... + \binom{d}{d} x^d \\ &= \binom{d}{1} x + ... + \binom{d}{d} x^d - \left[\binom{d-t}{1} x + ... + \binom{d-t}{d-t} x^{d-t}\right] \\ &= (1+x)^d - (1+x)^{d-t} \\ &= (1+x)^{d-t}((1+x)^t - 1). \end{aligned} \tag{2}$$
Hence, $DW(G,x) = (1+x)^{2(n+m-2)}((1+x)^{(m-2)(n-2)} - 1)$. ■

The $m$-book graph $B_m$ is defined as the graph Cartesian product $S_m \Box P_2$ , where $S_m$ is a star graph of $m+1$ vertices and $P_2$ is the path graph . The generalization of book graph is called stacked book graph $B_{m,n}$ of order $(m,n)$ and defined as the graph Cartesian product $S_m \Box P_n$, where $S_m$ is a star graph of $m+1$ vertices and $P_n$ [3, 5].

**Theorem 2.16.** Let $G \cong B_m$ be a book graph of $2(m+1)$ vertices, where $m \geq 2$. Then, $DW(G,x) = (1+x)^{2m}(x^2+2x)$. Furthermore, $Z_{dw}(G) = \{0, -1, -2\}$.

**Proof:** Let $G \cong B_m$ be a book graph of $2(m+1)$ vertices, where $m \geq 2$. We know that $\gamma_{dn}(G) = 1$ and there are two minimum downhill dominating sets of size one. Let $n = 2(m+1)$, then $dw(G,1) = \binom{n}{1} - \binom{n-2}{1} = 2$. For $i = 2, 3, ..., n-2$. Every downhill dominating set of size $i$ must be contains at least one of the two vertices which of degree $m$, so

$$dw(G,2) = \binom{n}{2} - \binom{n-2}{2}, ..., dw(G,n-2) = \binom{n}{n-2} - \binom{n-2}{n-2}.$$

Now, any $n-1$ and $n$ vertices are downhill dominating set. $dw(G,n-1) = \binom{n}{n-1}$ and $dw(G,n) = \binom{n}{n}$. Thus, we get

$$\begin{aligned} DW(G,x) &= \left[\binom{n}{1} - \binom{n-2}{1}\right] x + ... + \left[\binom{n}{n-2} - \binom{n-2}{n-2}\right] x^{n-2} + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n \\ &= \binom{n}{1} x + \binom{n}{2} x^2 + ... + \binom{n}{n} x^n - \left[\binom{n-2}{1} x + \binom{n-2}{2} x^2 + ... + \binom{n-2}{n-2} x^{n-2}\right] \\ &= (1+x)^n - (1+x)n - 2 \\ &= (1+x)^{n-2}((1+x)^2 - 1). \end{aligned}$$

Hence, $DW(G,x) = (1+x)^{2m}(x^2 + 2x)$. ∎

The generalization of Theorem 2.16 is obtained in Theorem 2.17

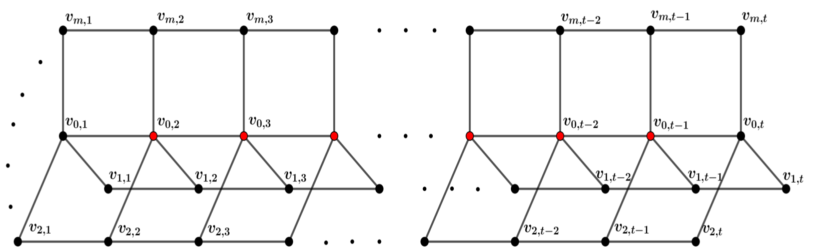

Figure 1: Stacked book graph $B_{m,t}$.

**Theorem 2.17.** Let $G \cong B_{m,t}$ be stacked book graph of $t(m+1)$ vertices as in Figure 1, where $m \geq 2$ and $t \geqslant 3$. Then, $DW(G,x) = (1+x)^{tm+2}((1+x)^{t-2} - 1)$.

**Proof:** Let $G \cong B_{m,t}$ be stacked book graph of $t(m+1)$ vertices, where $m \geq 2$ and $t \geqslant 3$ as in Figure 1. It has $t-2$ minimum downhill dominating sets of size one (red vertices). Suppose that $n = t(m+1)$, then

$$dw(G,1) = \binom{n}{1} - \binom{n-t+2}{1} = t-2.$$

For $i = 2, 3, ..., n-t+2$. Every downhill dominating set of size $i$ must be contains at least one vertex of the degree $m+2$ (red vertices).

$$dw(G,2) = \binom{n}{2} - \binom{n-t+2}{2}, ..., dw(G, n-t+2) = \binom{n}{n-t+2} - \binom{n-t+2}{n-t+2}.$$

For $i = n-t+3, ..., n$. Any $i$ vertices are downhill dominating set of size $i$.

$$dw(G, n-t+3) = \binom{n}{n-t+3}, ..., dw(G,n) = \binom{n}{n}.$$

Thus, we get

$$\begin{aligned} DW(G,x) &= \left[\binom{n}{1} - \binom{n-t+2}{1}\right]x + \left[\binom{n}{2} - \binom{n-t+2}{2}\right]x^2 + ... + \binom{n}{n}x^n \\ &= \binom{n}{1}x + ... + \binom{n}{n}x^n - \left[\binom{n-t+2}{1}x + ... + \binom{n-t+2}{n-t+2}x^{n-t+2}\right] \\ &= (1+x)^n - (1+x)^{n-t+2} \\ &= (1+x)^{n-t+2}((1+x)^{t-2} - 1). \end{aligned}$$

Hence, $DW(G,x) = (1+x)^{tm+2}((1+x)^{t-2} - 1)$. ∎

**Proposition 2.18.** Let $G \cong C_n \square P_m$ be the graph with $nm$ vertices, where $m \geq 3$. Then

$$DW(G,x) = (1+x)^{2n}((1+x)^{n(m-2)} - 1).$$

**Proof:** Let $G \cong C_n \square P_m$ be the graph with $nm$ vertices, where $m \geq 3$. The graph has $n(m-2)$ minimum downhill dominating sets of size one. Then,

$$dw(G,1) = \binom{nm}{1} - \binom{2n}{1} = n(m-2).$$

For $i = 2, 3, ..., 2n$. Every downhill dominating set in $G$ must be contains at least one minimum downhill dominating set.

$$dw(G,2) = \binom{nm}{2} - \binom{2n}{2}, ..., dw(G,2n) = \binom{nm}{2n} - \binom{2n}{2n}.$$

For, $i = 2n+1, ..., nm$. Any $i$ vertices are downhill dominating set of size $i$. Then.

$$dw(G, 2n+1) = \binom{nm}{2n+1}, ..., dw(G, nm) = \binom{nm}{nm}.$$

Thus, we get

$$DW(G,x) = \binom{nm}{1}x + \binom{nm}{2}x^2 + ... + \binom{nm}{nm}x^{nm} - \left[\binom{2n}{1}x + \binom{2n}{2}x^2 + ... + \binom{2n}{2n}x^{2n}\right].$$

Hence, $DW(G, x) = (1+x)^{2n}((1+x)^{n(m-2)} - 1)$. ∎

**Corollary 2.19.** For any connected regular graph $G$ with $n$ vertices and $m \geq 3$, then $DW(G\Box P_m, x) = (1+x)^{2n}((1+x)^{n(m-2)} - 1)$.

Finally, we get two general results for the downhill domination polynomial of graphs.

**Theorem 2.20.** Let $G$ be a graph of $n$ vertices with $\gamma_{dn}(G) = 1$ and has $r$ minimum downhill dominating sets. Then, $DW(G, x) = (1+x)^n - (1+x)^{n-r}$.

**Proof:** Let $G$ be a graph of $n$ vertices with $\gamma_{dn}(G) = 1$ and has $r$ minimum downhill dominating sets. Then, we have

$$dw(G, 1) = \binom{n}{1} - \binom{n-r}{1} = r.$$

For $i = 2, ..., n-r$. Every downhill dominating set of size $i$ must be contains at least one minimum downhill dominating set. Then

$$dw(G, 2) = \binom{n}{2} - \binom{n-r}{2}, ..., dw(G, n-r) = \binom{n}{n-r} - \binom{n-r}{n-r}.$$

For $i = n-r+1, ..., n$. Any $i$ vertices are downhill dominating set of size $i$. Then

$$dw(G, n-r+1) = \binom{n}{n-r+1}, ..., dw(G, n) = \binom{n}{n}.$$

Thus, we get

$$\begin{aligned} DW(G,x) &= \left[\binom{n}{1} - \binom{n-r}{1}\right]x + ... + \left[\binom{n}{n-r} - \binom{n-r}{n-r}\right]x^{n-r} \\ &+ \binom{n}{n-r+1}x^{n-r+1} + ... + \binom{n}{n}x^n \end{aligned}$$

$$= \binom{n}{1}x + ... + \binom{n}{n-r}x^{n-r} + ... + \binom{n}{n}x^n - \left[\binom{n-r}{1}x + ... + \binom{n-r}{n-r}x^{n-r}\right]$$
$$= (1+x)^n - (1+x)^{n-r}$$

■

By using Matlab 2017, we have write code to calculate the downhill domination roots of the graph of $n$ vertices with $\gamma_{dn}(G) = 1$ and has $r$ minimum downhill dominating sets.

In Figure 2, we present the behaviors and positions of the downhill domination roots of all the graphs of $n \leq 20$ vertices with $\gamma_{dn}(G) = 1$ and has $r \leq 20$ minimum downhill dominating sets.

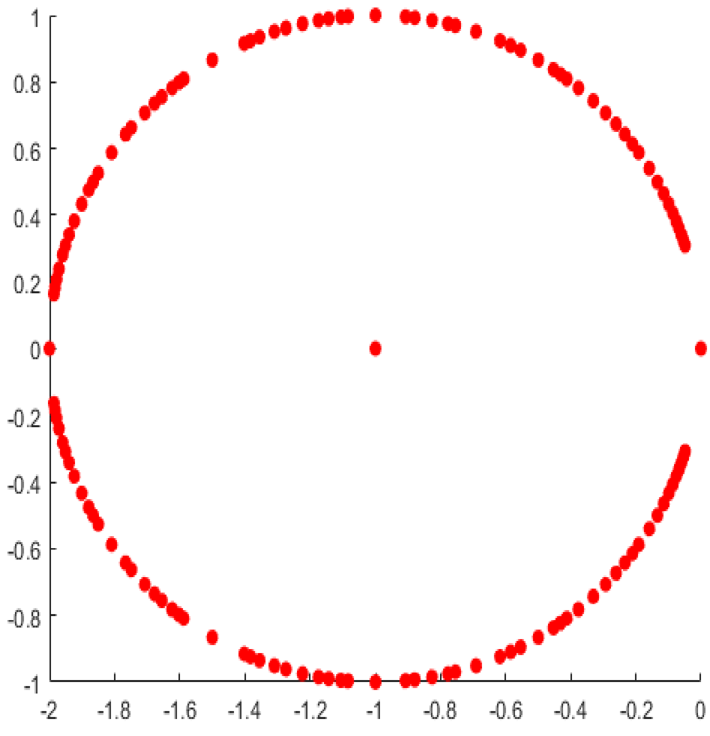


Figure 2: The positions of downhill domination roots of all the possibilities of the graphs of $n \leq 20$ vertices with $\gamma_{dn}(G) = 1$ and has $r \leq 20$ minimum downhill dominating sets

In the same way in Figure 3, we present the behaviors and positions of the downhill domination roots of all the possibilities of the graphs of $n \leq 100$ vertices with $\gamma_{dn}(G) = 1$ and has

$r \leq 100$ minimum downhill dominating sets.

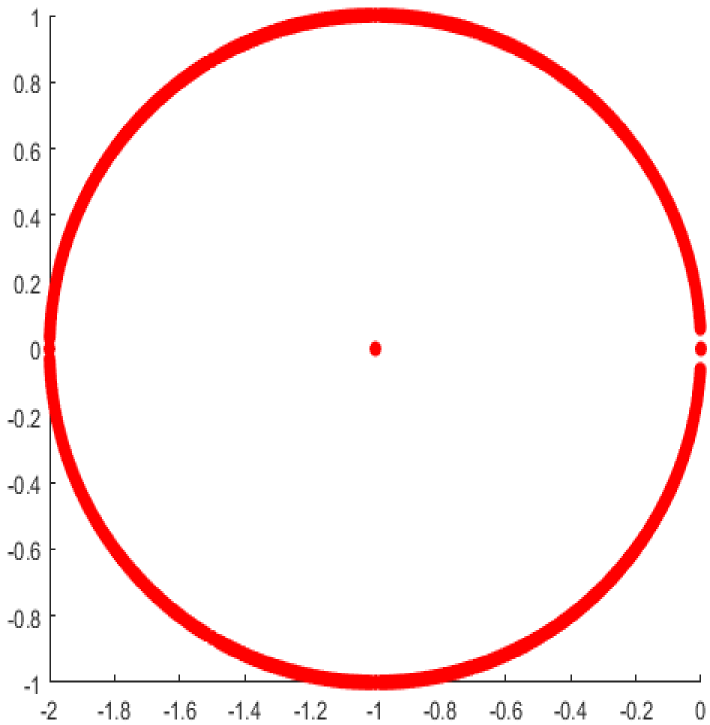


Figure 3: The positions of downhill domination roots of all the possibilities of the graphs of $n \leq 100$ vertices with $\gamma_{dn}(G) = 1$ and has $r \leq 100$ minimum downhill dominating sets

**Theorem 2.21.** Let $G$ be a graph of $n$ vertices with unique minimum downhill dominating set with size $s$. Then $DW(G, x) = x^s(1 + x)^{n-s}$.

**Proof:** Let $G$ be a graph of $n$ vertices with unique minimum downhill dominating set with size $s$. This means, $dw(G, s) = 1$. Then, for $i = s + 1, ..., n$, there are $\binom{n-s}{i-s}$ downhill dominating set of size $i$. Thus, we get

$$\begin{aligned} DW(G,x) &= \binom{n-s}{0}x^s + \binom{n-s}{1}x^{s+1} + ... + \binom{n-s}{n-s}x^n \\ &= x^s\left[\binom{n-s}{0}x^0 + \binom{n-s}{1}x + ... + \binom{n-s}{n-s}x^{n-s}\right] \\ &= x^s(1+x)^{n-s}. \end{aligned}$$

∎

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