International Journal of Applied Graph Theory
Vol.3, No.2 (2019), 29 - 38.
$ISSN(Online) : 2456 - 7884$

Pathos Square Graph of a Tree

H. M. Nagesh
Department of Science and Humanities
PES University - Electronic City Campus
Bangalore - 560 100, India.
nageshhm@pes.edu

Abstract

Let T be a tree of order n, $n \geq 2$. A *pathos square graph* of T, written $PG^2(T)$, is a graph whose vertices are the vertices and paths of a pathos of T, with two vertices of $PG^2(T)$ adjacent whenever the distance between the corresponding vertices of T is at most two; or the corresponding paths $P_i^{'}$ and $P_j^{'}$, $i \neq j$ of a pathos of T have a vertex in common; or one corresponds to the path $P^{'}$ and the other to a vertex v of T and $P^{'}$ begins (or ends) at v such that v is a pendant vertex. For this class of graphs we discuss the planarity; outerplanarity; maximal outerplanarity; minimally nonouterplanarity; Eulerian; and Hamiltonian properties.

Key words: Crossing number, inner vertex number, pathos, path number.
2010 Mathematics Subject Classification : 05C05, 05C45

1 Introduction

For graph theoretic terminology we refer to Harary [1]. There are many graph operators (or graph valued functions) with which one can construct a new graph from a given graph, such as the line graph, the total graph, and their generalizations. One such graph operator is called the *square graph*. This was introduced by Ross et al. in [3], and studied in [6].

Let $G = (V, E)$ be a graph of order n, $n \geq 2$. The *square* of G, written G^2, is that graph having the same vertex set as G, where two vertices are adjacent in G^2 if the distance between these two vertices in G is at most two.

An example of a graph G and its square G^2 is given in Figure.1.

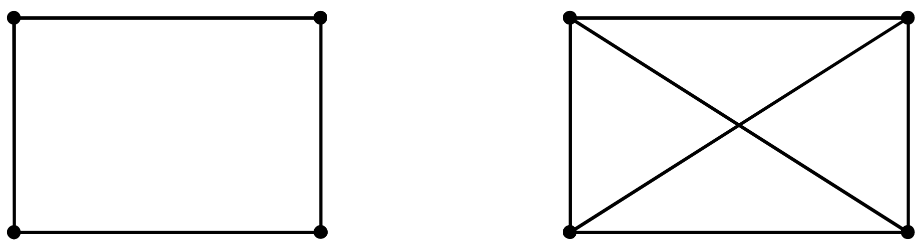

Figure.1

$*$ Corresponding Author: H. M. Nagesh
Ψ Received on March 27, 2019 / Revised on June 01, 2019 / Accepted on June 02, 2019

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G, with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have a vertex in common. The concept of $pathos$ of a graph G was introduced by Harary [2] as a collection of minimum number of edge disjoint open paths whose union is G. The *path number* of a graph G is the number of paths in any pathos. The path number of a tree T equals k, where $2k$ is the number of odd degree vertices of T.

Muddebihal et al. in [5] extended the concept of pathos of graphs to trees there by introducing a graph operator called a *pathos line graph* of a tree T. A *pathos line graph* of a tree T, written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T have a vertex in common or the edge lies on the corresponding path of the pathos.

Figure.2 gives an example of a tree along with pathos (indicated by dotted lines) and its pathos line graph.

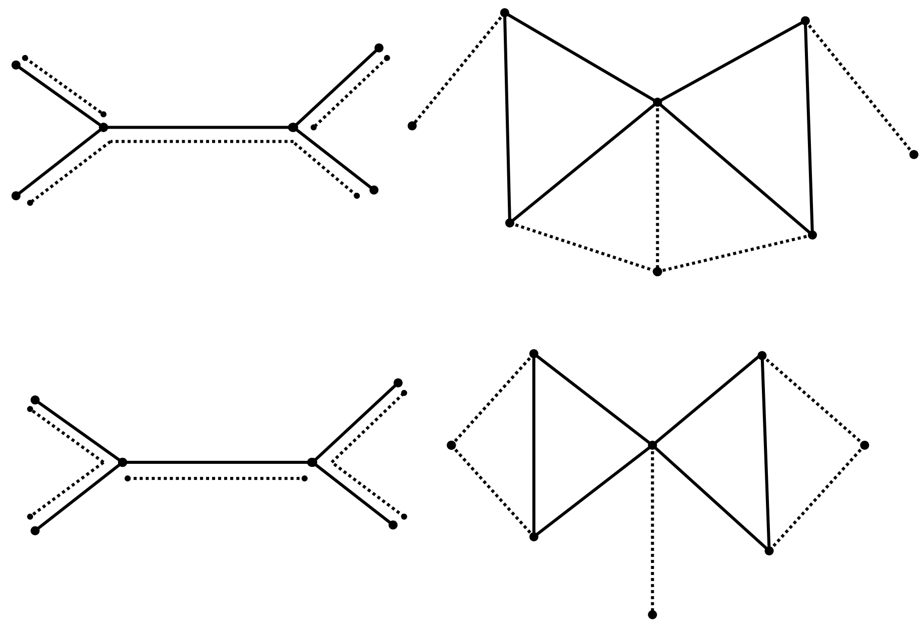

Figure 2

A *pathos vertex* of $PL(T)$ is a vertex corresponding to the path of a pathos of T. For example, the trees (on the left) of Figure.2 contains three paths of pathos, say $P_1^{'}, P_2^{'}$, and $P_3^{'}$. Thus $P_1^{'}, P_2^{'}$, and $P_3^{'}$ are the *pathos vertices* of the corresponding pathos line graph $PL(T)$ (on the right) of Figure.2.

Motivated by the studies above, we define a new graph operator called a *pathos square graph* of a tree.

2 Preliminaries

A $graph$ $G = (V, E)$ is a pair, consisting of some set V, the so-called *vertex set*, and some subset E of the set of all 2-element subsets of V, the *edge set*. We write $x = (p, q)$ and say that p and q are *adjacent vertices* (sometimes denoted p adj q). The $degree$ of a vertex v in G, denoted by $deg(v)$, is the number of edges of G incident with v, each loop counting as two edges. We denote by $\Delta(G)$ the maximum degree of the vertex of G. A *pendant vertex* is a vertex of degree 1 and an *internal vertex* is a vertex of degree at least 2.

A *planar graph* is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their end vertices. In other words, it can be drawn in such a way that no edges cross each other. Such a drawing is called a *plane graph* or *planar embedding of the graph*. If a planar graph G is embeddable in the plane so that all the vertices are on the boundary of the exterior region, then G is said to be $outerplanar$.

An outerplanar graph G is *maximal outerplanar* if no edge can be added without losing outerplanarity. For a planar graph G, the *inner vertex number* $i(G)$ is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of G in the plane. The least number of edge crossings of a graph G, among all planar embeddings of G, is called the *crossing number* of G and is denoted by $\mathrm{cr}(G)$.

2.1 Definition of $PG^2(T)$

Let T be a tree of order n, $n \geq 2$. A *pathos square graph* of T, written $PG^2(T)$, is a graph whose vertices are the vertices and paths of a pathos of T, with two vertices of $PG^2(T)$ adjacent whenever the distance between the corresponding vertices of T is at most two; or the corresponding paths P_i' and P_j' $(i \neq j)$ of a pathos of T have a vertex in common; or one corresponds to the path P' and the other to a vertex v of T and P' begins (or ends) at v such that v is a pendant vertex.

See Figure.3 for an example of a tree along with pathos (indicated by dotted lines) and its pathos square graph.

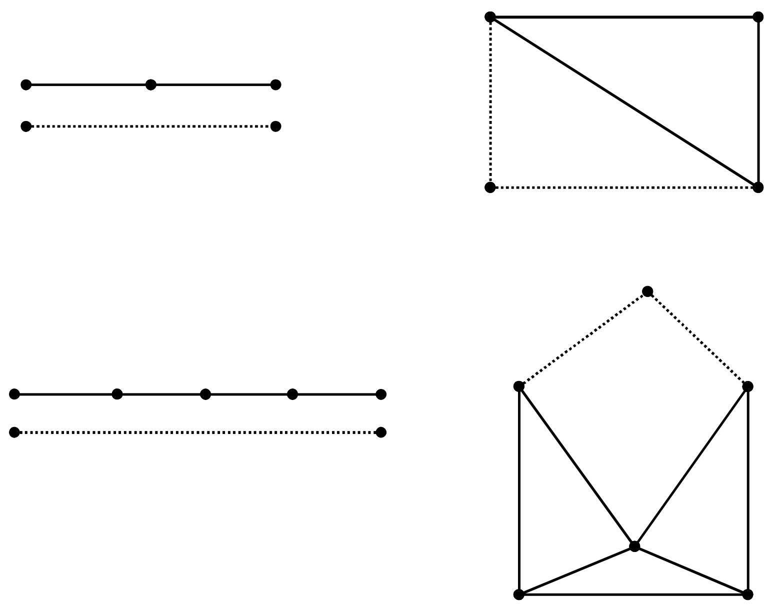

Figure 3

Note that there is freedom in marking the paths of a pathos of a tree T in different ways, provided that the path number k of T is fixed. For example, consider the marking of the paths of pathos of trees (on the left) of Figure.2, where $k = 3$. Therefore, we conclude that since the order of marking of the paths of a pathos of a tree is not unique, the corresponding pathos square graph is also not unique. This obviously raises the question of the existence of "unique" pathos square graph.

One can easily check that if the path number of a tree is exactly one, i.e., k=1, then the corresponding pathos square graph is unique. Since path number of a path P_n on $n \geq 2$ vertices is one, we can speak of "the" pathos square graph only for paths, i.e., the pathos square graph of a path is unique. Furthermore, one can also observe easily that for different ways of marking of the paths of a pathos of a star graph $K_{1,n}$ on $n \geq 3$ vertices, the corresponding pathos square graphs are isomorphic.

In this paper we look at some properties of $PG^2(T)$. For this class of graphs we also discuss the planarity; outerplanarity; maximal outerplanarity; minimally nonouterplanarity; Eulerian; and Hamiltonian properties of these graphs.

3 Properties of pathos square graphs

In this section we study certain properties of pathos square graph.

Property 3.1. *A pathos square graph does not contain any cut-vertex. Hence it is always a block.*

Property 3.2. *Let $e = (v_1, v_2)$ be an edge in T such that $deg(v_1) = 1$ and $deg(v_2) > 1$. Then $deg(v_1)$ in $PG^2(T)$ equals $deg(v_2) + 1$.*

Property 3.3. *Let $e = (v_1, v_2)$ be an edge in T such that $deg(v_1) > 1$ and $deg(v_2) > 1$. Then $deg(v_1)$ (or $deg(v_2)$) in $PG^2(T)$ equals $deg(v_1) + deg(v_2) - 1$.*

Property 3.4. *If T is a path of order two and three, then the inner vertex number of $PG^2(T)$ is zero, that is, $i(PG^2(T)) = 0$. If $T = P_4$, then $i(PG^2(T)) = 1$. Furthermore, if T is a path of order $(2n + 3)$ and $(2n + 4)$ for $n \geq 1$, then $i(PG^2(T)) = n + 1$.*

Property 3.5. *Let $T = \{v_1, v_2, \ldots, v_n\}$ be a path of order $n \geq 3$. Then the degree of the vertices v_1 and v_n in $PG^2(T)$ is three.*

Property 3.6. *Let T be a tree of order n $(n \geq 3)$. Then the number of edges whose end-vertices are the pathos vertices in $PG^2(T)$ is at most $\frac{k(k-1)}{2}$, where k is the path number of T. In particular, if T is a star graph $K_{1,n}$ on $n \geq 3$ vertices, then the number of edges whose end-vertices are the pathos vertices in $PG^2(T)$ is exactly $\frac{k(k-1)}{2}$, i.e., in a pathos square graph of a star graph, the pathos vertices are pairwise adjacent.*

While defining any class of graphs, it is desirable to know the order and size of each. In order to find the order and size of $PG^2(T)$, we first find the order and size of the square of a tree.

Let $V(T) = \{v_1, v_2, \ldots, v_n\}$ be vertex set of T and let α and β be number of internal and pendant vertices of T, respectively. Clearly, $n = \alpha + \beta$, and thus $\alpha = n - \beta$.

Our next two results give the order and size of the square of a tree. The proof is straightforward, so we omit it.

Proposition 3.7. *Let T be a path P_n on $n \geq 3$ vertices and $\alpha_1, \alpha_2, \ldots, \alpha_{n-2}$ be internal vertices of T. Then the size of T^2 equals $\sum_{i=1}^{n-2} \{deg(\alpha_i)\} + 1$.*

Proposition 3.8. *Let T be a tree (except a path) on $n \geq 3$ vertices, and let $\alpha_1, \alpha_2, \ldots, \alpha_{n-\beta}$ be internal vertices of T and s be the degree of each internal vertex of T. For each internal vertex, let $b_l = l$ for $(1 \leq l \leq s)$. Then the size of T^2 equals $\sum_{i=1}^{n-\beta} \sum_{j=1}^{s} a_i b_j$, where $a_i = 1$ for $1 \leq i \leq n - \beta$.*

Our next result gives the number of pendant vertices in a tree T which is also essential while finding the size of $PG^2(T)$.

Proposition 3.9. Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. Then the number of pendant vertices in T is $2 + \sum_{deg(v)\geq 3} (deg(v) - 2)$.

Proof: Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. Let β be the number of pendant vertices in T. By the handshaking lemma, we have $\sum_{v\in T} deg(v) = 2(n-1) = 2n-2$.
$\Rightarrow -2 = \sum_{v\in T} deg(v) - 2n$
$\Rightarrow -2 = \sum_{v\in T} deg(v) - \sum_{v\in T} 2$
$\Rightarrow -2 = \sum_{v\in T}(deg(v) - 2)$. On taking the sum over the vertices of degree one and two, we get
$-2 = \sum_{deg(v)=1} (-1) + \sum_{deg(v)=2} (0) + \sum_{deg(v)\geq 3} (deg(v) - 2)$
$\Rightarrow -2 = -\beta + \sum_{deg(v)\geq 3} (deg(v) - 2)$
$\Rightarrow \beta = 2 + \sum_{deg(v)\geq 3} (deg(v) - 2)$. ∎

The following result gives the order and size of $PG^2(T)$.

Property 3.10. Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. Then the maximum number edges in $PG^2(T)$ is $\sum_{i=1}^{n-\beta}\sum_{j=1}^{s} a_i b_j + 2 + \sum_{deg(v)\geq 3} (deg(v) - 2) + \frac{k(k-1)}{2}$, where k is the path number, and α and β be the number of internal and pendant vertices of T, respectively. Moreover, this bound is sharp.

Proof: Let T be a tree with vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$. By definition, the order of $PG^2(T)$ equals the sum of vertices and the path number of T. Thus $V(PG^2(T)) = n + k$. The size of $PG^2(T)$ equals the sum of size of T^2; number of pendant vertices in T; and the number of edges whose end-vertices are the pathos vertices. By Property 3.6, Proposition 3.8, and Proposition 3.9, the maximum number edges in $PG^2(T)$ equals
$\sum_{i=1}^{n-\beta}\sum_{j=1}^{s} a_i b_j + 2 + \sum_{deg(v)\geq 3} (deg(v) - 2) + \frac{k(k-1)}{2}$. ∎

As an example, the tree in the following figure exhibits that the bound is sharp. Here the size of $PG^2(T)$ equals $(3 + 2 + 1) + 2 + (3 - 2) + 1 = 10$.

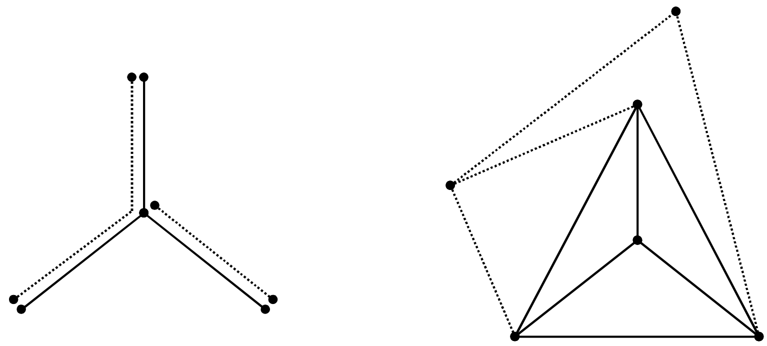

Figure 4: A tree and its pathos square graph

4 Characterization of $PG^2(T)$

4.1 Planar pathos square graphs

We now characterize the graphs whose $PG^2(T)$ is planar.

Theorem 4.1. A pathos square graph $PG^2(T)$ of a tree T is planar if and only if the number of pendant vertices in T is at most three.

Proof: Suppose $PG^2(T)$ is planar. Assume that the number of pendant vertices in T is at least four. Suppose it is four. Let $T = P_2 \times P_3 - 2e$, where e is an edge between the even degree vertices of $P_2 \times P_3$. Let $V(T) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be vertex set of T. By definition, $T^2 = K_6 \setminus \{(v_1, v_5), (v_1, v_6), (v_3, v_5), (v_3, v_6)\}$. Clearly the crossing number of T^2 is two, i.e., cr$(T^2) = 2$. Since $T^2 \subseteq PG^2(T)$, it contradicts the assumption that cr$(PG^2(T)) = 0$.

Conversely, suppose that the number of pendant vertices in T at most three. We consider the following two cases.
$Case$ 1: If $T = P_2$, then $PG^2(P_2) = K_3$, which is planar. If $T = P_3$, then $PG^2(P_2) \cong K_4 - e$, which is also planar. On the other hand, let T be a path on $n \geq 4$ vertices. Let $V(T) = \{v_1, v_2, \ldots, v_n\}$. Then (v_i, v_{i+1}) for $1 \leq i \leq n-1$ and (v_j, v_{j+2}) for $1 \leq j \leq n-2$, are the edges of T^2. Since the path number of T is one, say P' and P' is adjacent to both v_1 and v_n, the crossing number of $PG^2(T)$ becomes zero. Thus $PG^2(T)$ is planar.
$Case$ 2: Suppose now that the number of pendant vertices of T is three, i.e., $T \cong K_{1,3}$. Let $V(T) = \{v_1, v_2, v_3, v_4\}$. Then $T^2 = K_4$. Let $P(T) = \{P_1', P_2'\}$ be a pathos set of T such that P_1' lies on the arcs $(v_1, v_2), (v_2, v_3)$ and P_2' lies on (v_2, v_4). Then the pathos vertex P_1' is adjacent to v_2, v_3, P_2' and P_2' is adjacent to v_4. This shows that cr$(PG^2(T)) = 0$. This completes the proof. ∎

We now establish a characterization of graphs whose $PG^2(T)$ are outerplanar; maximal outerplanar; minimally nonouterplanar; and crossing number one.

Theorem 4.2. A pathos square graph $PG^2(T)$ of a tree T is outerplanar if and only if $\Delta(T) \leq 2$, for every vertex $v \in T$, and T contains exactly one vertex of degree two.

Proof: Suppose $PDP(T)$ is outerplanar. Assume that $\Delta(T) \leq 2$ and T contains two vertices of degree two. Then $T \simeq P_4$. By Case 1 of sufficiency of Theorem 4.1, cr$(PG^2(T)) = 0$ and by Property 3.4, $i(PG^2(T)) = 1$, a contradiction. On the other hand, if there exists a vertex of degree three in T. Then $T \simeq K_{1,3}$. By Case 2 of sufficiency of Theorem 4.1, cr$(PG^2(T)) = 0$, but $i(PG^2(T)) = 2$, again a contradiction.

Conversely, suppose that $\Delta(T) \leq 2$, for every vertex $v \in T$, and T contains exactly one vertex of degree two, i.e., $T \cong P_3$. By definition, $PG^2(T) \cong K_4 - e$, which is outerplanar. This completes the proof. ∎

Theorem 4.3. A pathos square graph $PG^2(T)$ of a tree T is maximal outerplanar if and only if T is either P_2 or P_3.

Proof: Suppose $PG^2(T)$ is maximal outerplanar. Assume that there exists a vertex of degree three in T. By Theorem 4.2, $PG^2(T)$ in nonouterplanar, a contradiction. On the other hand, if T is a path of order n $(n \geq 4)$, then Property 3.4 implies that $PG^2(T)$ is nonouterplanar, again a contradiction.

Conversely, suppose that T is P_2. Then $PG^2(T) = K_3$, which is maximal outerplanar. On the other hand, if T is P_3, then $PG^2(T) \cong K_4 - e$, which is also maximal outerplanar. This completes the proof. ∎

The following characterization of minimally nonouterplanar graphs is well known.

Theorem 4.4. (**V. R. Kulli** [4]) : A graph G is minimally nonouterplanar if and only if the inner vertex number of G is one, i.e., $i(G) = 1$.

Theorem 4.5. A pathos square graph $PG^2(T)$ of a tree T is minimally nonouterplanar if and only if T is P_4.

Proof: Suppose $PG^2(T)$ is minimally nonouterplanar. Assume that $T = P_5$. By Property 3.4, $i(PG^2(T)) = 2$, a contradiction.

Conversely, suppose that $T = P_4$. By Property 3.4, $i(PG^2(T)) = 1$, and thus Theorem 4.4 implies that $PG^2(T)$ is minimally nonouterplanar. This completes the proof. ∎

Theorem 4.6. For any tree T, $PG^2(T)$ does not have crossing number one.

Proof: We use contradiction. Suppose that $PG^2(T)$ has crossing number one. We consider the following two cases.
$Case$ 1: Suppose that $\Delta(T) \leq 2$. By Theorem 4.1, $PG^2(T)$ is planar, a contradiction.
$Case$ 2: Suppose that $\Delta(T) \geq 3$. If there exists a vertex of degree three in T, then Case 2 of Theorem 4.1 implies that cr$(PG^2(T)) = 0$, a contradiction. On the other hand, if there exists two vertices of degree three in T, then the necessity of Theorem 4.1 implies that cr$(T^2) = 0$, but cr$(PG^2(T)) > 1$, again a contradiction. Hence by all the cases above, cr$(PG^2(T)) \neq 1$. This completes the proof. ∎

4.2 Eulerian pathos square graphs

A $tour$ of a connected graph G is a closed walk that traverses each edge of G at least once, and an *Euler tour* one that traverses each edge exactly once (in other words, a closed Euler trail). A graph is $Eulerian$ if it admits an Euler tour.

We now investigate the Eulerian property of $PG^2(T)$. The following result is well known.

Theorem 4.7. (**F. Harary** [1]) : A connected graph G is Eulerian if and only if each vertex in G has even degree.

Theorem 4.8. Let T be a tree (except a path P_n) of order n $(n \geq 3)$. Then a pathos square graph $PG^2(T)$ of T is Eulerian if and only if the number of pendant vertices in T is two.

Proof: Suppose that $PG^2(T)$ is Eulerian. Assume that the number of pendant vertices in T is at least three. If there exists three vertices of degree one in T, i.e., $T = K_{1,3}$, then $PG^2(T)$ contains K_4 as an induced subgraph. By Theorem 4.7, $PG^2(T)$ is non-Eulerian, a contradiction.

Conversely, suppose that the number of pendant vertices in T (except a path P_n of order $n \geq 3$) is two. By definition, $PG^2(T) = K_3$, which is Eulerian. This completes the proof. ∎

4.3 Hamiltonian pathos square graphs

A *Hamiltonian cycle* is a cycle that visits each vertex exactly once (except for the vertex that is both the initial and end, which is visited twice). A graph that contains a Hamiltonian cycle is called a *Hamiltonian graph*.

We characterize the graphs whose $PG^2(T)$ is Hamiltonian.

Theorem 4.9. A pathos square graph $PG^2(T)$ of a tree T is Hamiltonian if T is a path of order $n, n \geq 3$.

Proof: Suppose that T is a path of order n, $n \geq 3$. Clearly, the path number of T is exactly one, say $P^{'}$. Let $V(T) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set T. Then $\{v_1, v_2, \ldots, v_n\} \cup P^{'}$ is the vertex set of $PG^2(T)$. In forming $PG^2(T)$, $P^{'}$ becomes a vertex adjacent to the vertices v_1 and v_2 of T^2. Also, the edges (v_i, v_{i+1}) and (v_i, v_{i+2}) for $1 \leq i \leq n-2$; and (v_{n-1}, v_n) for $n \geq 3$, exists in $PG^2(T)$. Clearly, there exist a cycle $P^{'}, v_1, v_2, \ldots, v_n, P^{'}$ containing all the vertices of $PG^2(T)$. Hence $PG^2(T)$ is Hamiltonian. ∎

Acknowledgement

The authors are grateful to the anonymous referee for giving valuable comments and suggestions.

References

[1] F. Harary, Graph Theory, Addison-Wesley, Reading, Mass (1969).

[2] F. Harary, Converging and packing in graphs-I, Annals of New York Academy of Science, 175 (1970), 198-205.

[3] Ian C Ross and F. Harary, The square of a tree, The Bell System Technical Journal, 39 (1960), 641-647.

[4] V. R. Kulli, On minimally nonouterplanar graphs, Proceeding of the Indian National Science Academy, 40 (1975), 276-280.

[5] M. H. Muddebihal and R. Chandrasekhar, On pathos line graph of a tree, National Academy of Science Letters, 24 (2001), 116-123.

[6] A. Mukhopadhyay, The square root of a graph, Journal of Combinatorial Series, 2 (1967), 290-295.