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Prime Labelling of Cycle Related Special Class of Graphs

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Abstract

Prime labelling originated with Entringer and was introduced by Tout, Dabboucy and Howalla [5]. A Graph $G = (V, E)$ is said to have a **prime labelling** if its vertices are labeled with distinct integers $1, 2, 3, \ldots, |V(G)|$ such that for each edge xy the labels assigned to x and y are relatively prime. A graph admits a prime labeling is called a prime graph. In this paper, we prove that $P_m(S_n)$, $T_n(C_m)$,$L(C_n)$, $D(W_n)$ and $C(L_n) \bigodot K_1$ are prime graphs.

Keywords: Prime labelling, Prime graph

2010 Mathematics Subject Classification : 05C78

1 Introduction

A simple graph $G = (V, E)$ is said to have **order** $|V|$ and **size** $|E|$. A graph G is said to have a prime labeling (or called prime) if its vertices are labelled with distinct integers $1, 2, 3 \ldots, |V(G)|$, such that for each edges $xy \in E(G)$, the labels assigned to x and y are relatively prime [2]. The following definitions and notations are used in main results.

Definition 1.1. A graph is obtained from a path P_m with vertex set $u_1, u_2, \ldots, u_m$ by joining all consecutive vertices by path P_n with vertex set $v_1, v_2, \ldots, v_n$ in such a way that merging v_1 with u_i and v_n with u_{i+1}, $1 \leq i \leq n-1$ and so on. Then $P_m(S_n)$, $\forall$ m,n is called as polygonal snake graph.

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Definition 1.2. The lotus inside a circle $L(C_n)$ is a graph obtained from the cycle $C_n; u_1, u_2, \ldots, u_n, u_1$ and the star $K_{1,n}$ with central vertex v and the end vertices $v_1, v_2, \ldots, v_n$ by joining each u_i to v_i and v_{i+1} $(mod\text{n})$ with residues $1, 2, \ldots, n$.

Definition 1.3. A m-wheel graph of size n can be composed of $mC_n + K_1$, that is, it consists of m cycles of size n, where all the vertices of the m cycles are connected to a common vertex v_0. When $m = 2$, we call it as double wheel graph.

Lemma 1.4. If G is a prime graph of order n, then the independence number $\beta(G) \geq \lfloor \frac{n}{2} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x

2 Prime Labeling of Cycle Related Special Class of Graphs

Theorem 2.1. The graph $P_m(S_n)$ is prime, $\forall m$, n

Proof: Let $G = P_m(S_n)$
Let $V(G) = \{v_{(n-1)(k-1)+i}/1 \leq k \leq m-1, 1 \leq i \leq n-1\} \bigcup \{v_{(n-1)(m-1)+1}\}$
Let $E(G) = \{v_{(n-1)(k-1)+i}v_{(n-1)(k-1)+i+1}/1 \leq k \leq m-1, 1 \leq i \leq n-1\}$
There are $(m-1)(n-2)+m$ vertices and $n(m-1)$ edges.
Define $f : V \longrightarrow \{1, 2, \ldots, (m-1)(n-2)+m\}$ by
$f(v_i) = i, i = 1, 2, \ldots, (m-1)(n-2)+m$
$gcd\ (f(v_i), f(v_{i+1})) = gcd(i, i+1) = 1, i = 1, 2, \ldots, (m-1)(n-2)+m$
$gcd\ (f(v_{(n-1)(k-1)+1}), f(v_{(n-1)k+1})) = gcd((n-1)(k-1)+1, (n-1)k+1) = 1, k = 1, 2, \ldots, m-1$
Hence, $P_m(S_n)$ is a prime graph,$\forall m$, n. ∎

Example 2.2.

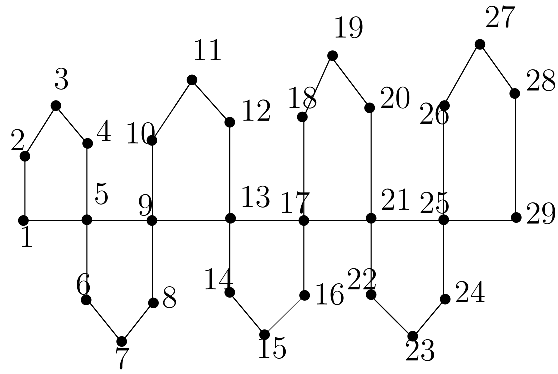

Figure 1:$P_8(S_5)$

Theorem 2.3. $T_n(C_m)$ is a prime

Proof: Let $G = T_n(C_m)$

Let $u_1, u_2, \ldots, u_m$ be the vertices of the cycle C_m and $u_1 = v_1, v_2, v_3, \ldots, v_{n+1}$ be the base vertices of the cycle C_3 and $w_1, w_2, \ldots, w_n$ be the upper vertices of the cycle C_3.

Let $V(G) = \{u_i, v_j, w_k / i = 1, 2, \ldots, m$ and $j = 2, 3, \ldots, n+1$ and $k = 1, 2, \ldots, n\}$

Let $E(G) = \{u_i u_{i+1}, u_m u_1 / i = 1, 2, \ldots, m-1\} \bigcup \{v_j v_{j+1}, u_1 = v_1 v_2 / j = 2, 3, \ldots, n\} \bigcup \{w_k v_k, w_k v_{k+1} / k = 1, 2, \ldots, n\}$

Define $f : V(G) \longrightarrow \{1, 2, \ldots, 2n+m\}$ by

$f(u_1 = v_1) = 1,$

$f(v_j) = 2j - 1, j = 2, 3, \ldots, n+1$

$f(w_k) = 2k, k = 1, 2, \ldots, n$

$f(u_i) = 2n + i, i = 2, 3, \ldots, m$

$gcd\ (f(u_i), f(u_{i+1})) = gcd(2n+i, 2n+i+1) = 1, i = 2, 3, \ldots, m-1$

$gcd\ (f(u_1), f(u_2)) = gcd(1, 2n+2) = 1$

$gcd\ (f(v_j), f(v_{j+1})) = gcd(2j-1, 2j+1) = 1, j = 1, 2, \ldots, n$

$gcd(f(w_k), f(v_k)) = gcd(2k, 2k-1) = 1, k = 1, 2, \ldots, n$

$gcd\ (f(w_k), f(v_{k+1})) = gcd(2k, 2k+1) = 1, k = 1, 2, \ldots, n$

Hence, $T_n(C_m)$ is a prime graph. ∎

Example 2.4.

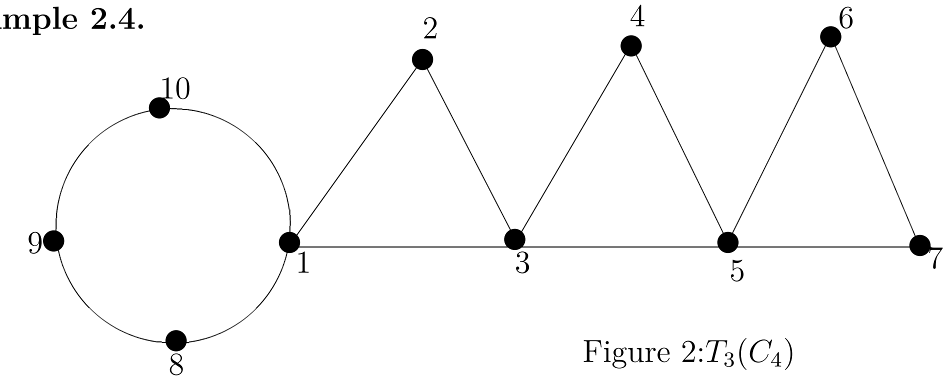

Figure 2:$T_3(C_4)$

Theorem 2.5. $L(C_n)$ is prime if and only if $n \not\equiv 1\ (mod\, 3)$, $n \geqslant 3$

Proof: Suppose $n \not\equiv 1\ (mod\, 3)$

Let $G = L(C_n)$

Let $V(G) = \{u_i, v_i, v/1 \leq i \leq n\}$

Let $E(G) = \{vv_i/1 \leq i \leq n\} \bigcup \{u_iu_{i+1}/1 \leq i \leq n-1\} \bigcup \{u_nu_1\} \bigcup \{v_iu_i/1 \leq i \leq n\} \bigcup \{v_{i+1}u_i/1 \leq i \leq n-1\} \bigcup \{v_1u_n\}$

There are $2n + 1$ vertices and $4n$ edges.

Define $f : V(G) \longrightarrow \{1, 2, \ldots, 2n + 1\}$ by $f(v) = 1$

$f(v_i) = 2i, i = 1, 2, \ldots, n$

$f(u_i) = 2i + 1, i = 1, 2, \ldots, n$

$gcd(f(v), f(v_i)) = gcd(1, 2i) = 1, i = 1, 2, \ldots, n$

$gcd(f(u_i), f(u_{i+1})) = gcd(2i + 1, 2i + 2) = 1, i = 1, 2, \ldots, n - 1$

$gcd(f(u_1), f(u_n)) = gcd(3, 2n + 1) = 1$

$gcd(f(v_i), f(u_i)) = gcd(2i, 2i + 1) = 1, i = 1, 2, \ldots, n$

$gcd\ (f(v_{i+1}),f(u_i)) = gcd(2i + 2, 2i + 1) = 1,i = 1, 2, \ldots, n - 1$

Hence $L(C_n)$ is a prime graph if $n \not\equiv 1\ (mod\, 3)$,$n \geqslant 3$

Let $n \equiv 1(mod\, 3)$

Suppose $L(C_n)$ is a prime

Let $v_1, v_2, \ldots, v_n$ be the outer rim vertices and let $u_1, u_2, \ldots, u_n$ be the interior vertices and u be the center of $L(C_n)$.

Note that there are n even numbers and $n+1$ odd numbers in $\{1, 2, \ldots, 2n+1\}$. First we label an even number to one of the outer rim vertices , say,v_1. As v is adjacent with all the interior vertices and $n \geq 3$, then remaining $n-1$ even numbers should be labelled to interior vertices only. Since v_1 is adjacent with two interior vertices, so we can label even numbers to $n-2$ remaining vertices only. But we have to label $n-1$ even numbers to interior vertices, this is not possible.Therefore, we cannot label any even number to the outer rim vertices. Hence, the only possibility is to label all the n even numbers to n interior vertices.

If $n \equiv 1(mod\, 3)$, then $2n+1 = 3l$.Now,$\{1, 2, \ldots, 2n+1\}$ contains $\frac{2n+1}{3}$ numbers which are multiples of 3. The rim vertices contain $\frac{n+2}{3}$ vertices whose labels are multiples of 3 and these vertices adjacent with $\frac{2(n+2)}{3}$ even numbered interior vertices, so these$\frac{2(n+2)}{3}$ vertices cannot ba labelled with multiples of 3.

Hence, these $\frac{2(n+2)}{3}$ vertices must be labelled with even numbers which are not multiples of 3. As there are n interior vertices, already $\frac{2(n+2)}{3}$ interior vertices are labelled with even numbers which are not multiples of 3, so the remaining interior vertices $= n - \frac{2(n+2)}{3} = \frac{n-4}{3}$ are to be labelled as multiples of 3. But $\{1, 2, \ldots, 2n+1\}$ contains $\frac{n-1}{3}$ even numbers which are multiples of 3 and they should be assigned among $\frac{n-4}{3}$ vertices, this is not possible. Hence, $L(C_n)$ is not a prime graph if $n \equiv 1(mod\, 3)$. ■

Example 2.6.

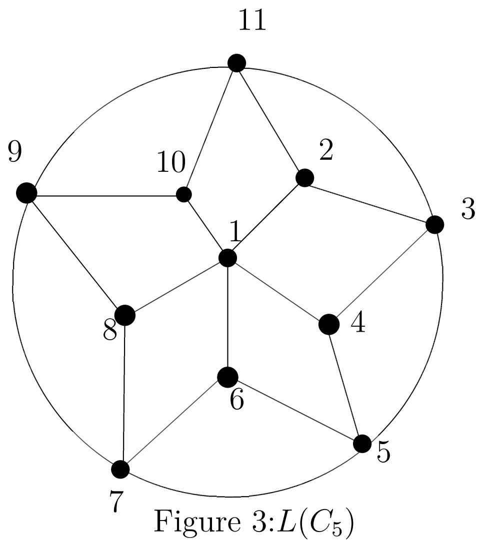

Figure 3:$L(C_5)$

Theorem 2.7. $D(W_n)$ is prime if and only if n is even

Proof: Let $G = D(W_n)$

Let $u_1, u_2, \ldots, u_n$ be the vertices of the inner wheel and $v_1, v_2, \ldots, v_n$ be the vertices of the outer wheel. Let w be the central vertex.

Let $V(G) = \{w, u_i, v_i / 1 \leq i \leq n\}$

Let $E(G) = \{wu_i, wv_i, u_j u_{j+1}, v_j v_{j+1}, u_1 u_n, v_1 v_n / 1 \leq i \leq n$ and $1 \leq j \leq n-1\}$

There are $2n+1$ vertices and $4n$ edges

Define $f : V(G) \longrightarrow \{1, 2, \ldots, 2n+1\}$ by $f(w) = 1$

suppose n is even .

Case1: If $n \equiv 0, 2 (mod\, 6)$

$f(u_i) = i + 1, i = 1, 2, \ldots, n$

$f(v_i) = n + i + 1, i = 1, 2, \ldots, n$

$gcd(f(w), f(u_i)) = gcd(1, i+1) = 1, i = 1, 2, \ldots, n$

$gcd(f(w), f(v_i)) = gcd(1, n+i+1) = 1, i = 1, 2, \ldots, n$

$gcd(f(u_i), f(u_{i+1})) = gcd(i+1, i+2) = 1, i = 1, 2, \ldots, n-1$

$gcd(f(u_1), f(u_n)) = gcd(1, n+1) = 1$

$gcd(f(v_1), f(v_n)) = gcd(n+2, 2n+1) = 1$

$gcd(f(v_i), f(v_{i+1})) = gcd(n+i+1, n+i+2) = 1, i = 1, 2, \ldots, n-1$

Case2: If $n \equiv\ 4 (mod\, 6)$

$f(u_i) = i + 3, i = 1, 2, \ldots, n$

$f(v_1) = 2, f(v_2) = 3$

$f(v_i) = n + i + 1, i = 3, 4, \ldots, n$

$gcd(f(u_i), f(u_{i+1}) = gcd(i+3, i+4) = 1, i = 1, 2, \ldots, n-1$

$gcd(f(u_1), f(u_n)) = gcd(4, n+3) = 1$

$gcd(f(v_1), f(v_n)) = gcd(2, 2n+1) = 1$

$gcd(f(v_2), f(v_3)) = gcd(3, n+4) = 1$

$gcd(f(v_i), f(v_{i+1})) = gcd(n+i+1, n+i+2) = 1, i = 3, 4, \ldots, n-1$

Hence,$D(W_n)$ is a prime graph if $n \equiv 0, 2, 4\ (mod\, 6)$

Conversely, if n is odd then $\beta(D(W_n)) = n - 1$ and $|\, D(W_n)\, | = 2n + 1$

Also, $\beta(D(W_n)) = n - 1 \leq \lfloor \frac{2n+1}{2} \rfloor = n$

By lemma 1.4, $D(W_n)$ is not a prime graph. ∎

Example 2.8.

Case(i):

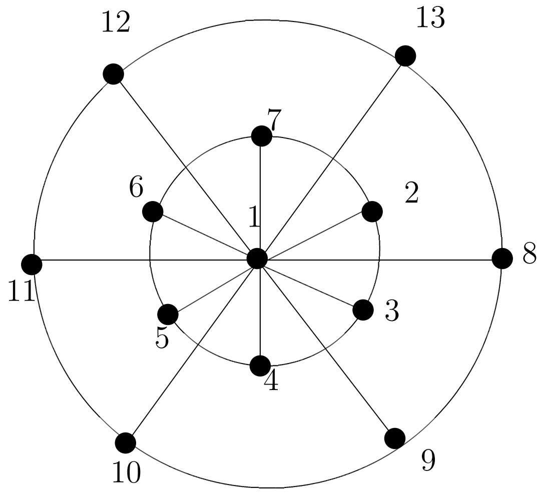

Figure 4: $D(W_6)$

Case(ii):

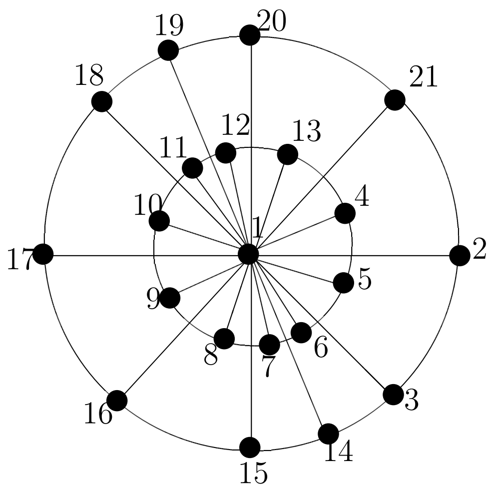

Figure 5: $D(W_{10})$

Theorem 2.9. $G = CL_n \odot K_1$ is a prime graph if $n \geq 3$

Proof: Let$V(G) = \{u_i, u_{i1}, v_i, v_{i1}/1 \leq i \leq n\}$

$E(G) = \{u_iu_{i+1}, v_iv_{i+1}/1 \leq i \leq n-1\} \bigcup \{u_nu_1, v_nv_1\} \bigcup \{u_iu_{i1}, v_iv_{i1}, u_iv_i/1 \leq i \leq n\}$

Then $|V(G)| = 4n$ and $|E(G)| = 5n$

Let $f : V(G) \longrightarrow \{1, 2, ..., 4n\}$ be defined as follows

$f(u_{11}) = 4, f(v_1) = 2, f(v_{11}) = 3$

$f(u_i) = 4i - 3, 1 \leq i \leq n$

$f(u_{i1}) = 4i - 2, 2 \leq i \leq n$

$f(v_i) = 4i - 1, 2 \leq i \leq n$

$f(v_{i1}) = 4i, 2 \leq i \leq n$

$gcd(f(u_i), f(u_{i+1})) = gcd(4i - 3, 4i + 1) = 1, 1 \leq i \leq n - 1$

$gcd(f(v_i), f(v_{i+1})) = gcd(4i - 1, 4i + 3) = 1, 1 \leq i \leq n - 1$

$gcd(f(u_i), f(u_{i1})) = gcd(4i - 3, 4i - 2) = 1, 2 \leq i \leq n$

$gcd\ (f(v_i), f(v_{i1})) = gcd(4i - 1, 4i) = 1, 2 \leq i \leq n$

$gcd\ (f(u_i), f(v_i)) = gcd(4i - 3, 4i - 1)$=1, $2 \leq i \leq n$

Hence, G is a prime graph. ∎

Example 2.10.

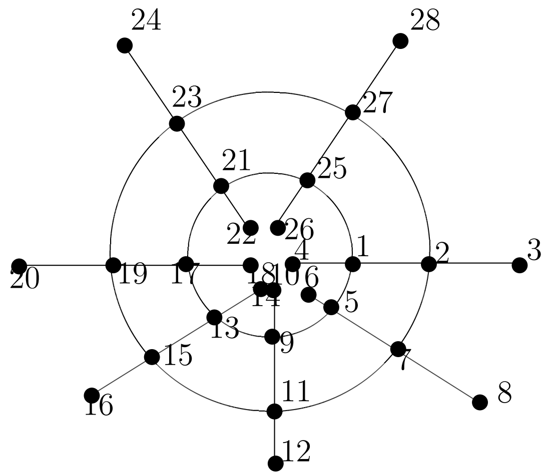

Figure 6:$CL_7 \odot K_1$

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