International Journal of Applied Graph Theory
Vol.3, No.2 (2019), 39 - 62.
$ISSN(Online) : 2456 - 7884$

Radio number of k^{th}-transformation graphs of a Path

B. Sooryanarayana
Department of Mathematics
Dr. Ambedkar Institute of Technology
Bengaluru, Karnataka, India.
dr_bsnrao@dr-ait.org
Narayani R. Shankar
Department of Mathematics
Maharani Lakshmi Ammanni College for Women
Bengaluru, Karnataka, India.
rsnarayani@gmail.com
Yogalakshmi S.
Department of Mathematics
East West Institute of Technology
Bengaluru, Karnataka, India.
yogalakshmis82@gmail.com

Abstract

Let $G = (V, E)$ be a finite and simple graph. A radio labeling of a graph G is a function $f : V(G) \to \{1, 2, \ldots, l\}$ such that $|f(u) - f(v)| \geq 1 + diam(G) - d_G(u, v)$ for every pair of vertices $u, v \in V(G)$, where $diam(G)$ is the diameter of G and $d_G(u, v)$ is the distance between u and v in G. The radio number of a graph G, denoted by $rn(G)$, is the smallest such integer l such that G admits a radio labeling. Let $T(G)$ be the total graph of G and α, β be two elements of $V(T(G))$. Then, the associativity of α and β is assigned as $+$ if the $d_{T(G)}(\alpha, \beta) \leq k$, otherwise it is assigned as $-$. Let xyz be a 3-permutation of the set $\{+, -\}$. The pair α and β is said to correspond to one of x, y or z of xyz if α and β are either both in $V(G)$ or in $E(G)$, or one of them is in $V(G)$ and the other in $E(G)$ respectively. The transformation graph $(G^{xyz})_k$ of G is the graph whose vertex set is $V(G) \cup E(G)$, where two of its vertices, α and β, are adjacent if and only if their associativity in G is consistent with the corresponding element of xyz. In this paper, we completely determine the radio number of transformation graphs of a path, for the case $k = 2$.

Key words: Radio-labeling, Radio number, Radio graceful graphs

2010 Mathematics Subject Classification : 05C12, 05C15, 05C78

$*$ Corresponding Author: B. Sooryanarayana
Ψ Received on March 03, 2019 / Revised on July 02, 2019 / Accepted on July 03, 2019

1 Introduction

The graphs considered in this paper are finite, simple, and undirected. The terms not defined here may be found in [1, 7]. Let $d_G(u,v)$ or simply $d(u,v)$ denote the distance between the vertices u and v in G. The maximum of all these distances in G is called the diameter of G and is denoted by $diam(G)$. For each vertex v, the i^{th}-open neighbourhood of v is the set $N_i(v) = \{u \in V(G) : d(v,u) = i\}$. Let $P_G(i) = \{\{x,y\} : d_G(x,y) = i\}$. Each element in $P_G(i)$ is called a k_i-pair of G. Further, a sequence $x_1, x_2, \ldots, x_j$ of vertices of G is called a $P_G(i)$ path if $\{x_i, x_{i+1}\} \in P_G(i)$ for each $1 \leq i \leq j-1$. The k_i-degree of a vertex v, denoted by $deg_{k_i}(v)$, is defined as $deg_{k_i}(v) = |N_i(v)|$. Let $K(G)$ be a weighted complete graph of order $|V(G)|$ with weight $w(xy) = d_G(x,y)$. Then $\langle P_G(i) \rangle$ is an edge induced subgraph of $K(G)$.

Labeling of a connected graph G is defined as an injection $f : V(G) \to \{1, 2, \ldots, l\}$. Radio labeling is one such labeling of G with a special property, $|f(u) - f(v)| \geq 1 + diam(G) - d_G(u,v)$, for every $u, v \in V(G)$. The radio number of f, denoted by $rn(f)$, is the maximum integer assigned to a vertex of G by the labeling f. Radio number of G is the $min\{rn(f)\}$ over all such radio labelings f of G. A radio labeling f of G is said to be a minimal radio labeling of G if $rn(f) = rn(G)$.

The channel assignment problem introduced by W. K. Hale et al [6] in 1980 paved way to the concept of radio labeling. This is applied in restricting FM radio channels [3] to eliminate unwanted disturbance or noise, consequently improving the clarity. The objective of the problem is to obtain the lower bound of the maximum frequencies of all the radio stations under the network. The concept of radio labeling was introduced by G. Chartrand, D. Erwin, P. Zhang, and F. Harary in [2]. Various authors have obtained the radio number of several networks in [4, 8, 9, 10, 11, 12, 13, 14, 16, 17].

In 2005, D. D. F. Liu and X. Zhu [11] calculated the radio number for paths and cycles. The radio labeling of square of paths was later obtained by D. D. F. Liu and M. Xie in [10]. The results of [10] have been generalised to k^{th} power of a path by P. Devadasa Rao, B. Sooryanarayana and Chandru Hegde in [5]. We recall the following result from [5] for immediate reference.

Theorem 1.1 ([5])**.** For any two positive integers n and k with $2 \leq k \leq n-2$,

$$rn(P_n^k) = \begin{cases} 2kp^2 + 2, & \text{if } (m = 0) \text{ or } (m = 1) \text{ and } n < 4k+1 \\ 2kp^2 + 3, & \text{if } (m = 1) \text{ and } n \geq 4k+1 \\ 2kp^2 + 2kp + m + 1, & \text{if } 2 \leq m \leq k \\ 2kp^2 + 2kp + m, & \text{if } m = k+1 \\ 2kp^2 + 4kp + 2k + 2, & \text{if } k+2 \leq m \leq 2k-1 \end{cases}$$

where $p = \lfloor \frac{n}{2k} \rfloor$ and $m = n - 2kp$.

Theorem 1.2 (D. D. F. Liu and Xuding Zhu [11])**.** For any integer $n \geq 2$,

$$rn(P_n) = \begin{cases} 4, & \text{if } n = 3 \\ 2k^2 + 3, & \text{if } n = 2k+1 \text{ and } k > 1 \\ 2k(k-1) + 2, & \text{if } n = 2k \end{cases}$$

Theorem 1.3 (D. D. F. Liu and Xuding Zhu [11])**.** Let C_n be the n-vertex cycle, $n \geq 3$. Then

$$rn(C_n) = \begin{cases} \frac{n-2}{2}\phi(n) + 2, & \text{if } n \equiv 0, 2 (mod\, 4) \\ \frac{n-1}{2}\phi(n) + 1, & \text{if } n \equiv 1, 3 (mod\, 4) \end{cases}$$

where $\phi(n) = \begin{cases} k+1, & \text{if } n = 4k+1 \\ k+2, & \text{if } n = 4k+r \text{ for some } r = 0, 2, 3 \end{cases}$

In 2001, Wu and Meng [15] introduced a new graphical transformation which generalizes the concept of total graph. In this paper, we generalise the concept of transformation graphs introduced by Wu and Meng in [15] and determine the radio number of its particular case for the path P_n.

2 Generalised Transformation Graph

Let $G = (V, E)$ be a finite and simple graph and α, β be two elements of $V(G) \cup E(G)$. Let $T(G)$ be the total graph of G. If $d_{T(G)}(\alpha, \beta) \leq k$, then the associativity of α and β in G is taken as $+$, else it is counted as $-$. Let xyz be a 3-permutation of the set $\{+, -\}$. There exists eight transformation graphs of G corresponding to the eight distinct 3-permutations of $\{+, -\}$. A pair of vertices $\alpha, \beta \in T(G)$ is said to correspond to x if both $\alpha, \beta \in V(G)$, y if both $\alpha, \beta \in E(G)$ and z if one is in $V(G)$ and the other is in $E(G)$. The k^{th} transformation graph $(G^{xyz})_k$ of G is the graph whose vertex set is $V(G) \cup E(G)$, where two of its vertices α and β are adjacent if and only if their associativity in G is consistent with the corresponding element of xyz. In particular, $(G^{xyz})_1 = G^{xyz}$, G^{+++} is exactly the total graph $T(G)$, and G^{---} is the complement of $T(G)$. The other six graphs G^{++-} and G^{--+}; G^{+-+} and G^{-+-}; and G^{-++} and G^{+--} form three pairs of complementary graphs. For example, the graph $(P_n^{xyz})_k$ is shown in Figure 1

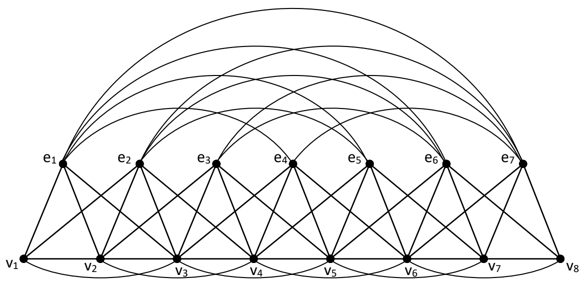

Figure 1: The graph $(P_8^{+-+})_3$.

Remark 2.1. From the definition of radio labeling it follows immediately that

$$rn(G) \geq |V(G)|. \tag{1}$$

3 Main Results

Since the transformation graph $(P_n^{+++})_2$ is isomorphic to P_{2n-1}^4 for $n \geq 3$ and isomorphic to C_3 if $n = 2$, from Theorem 1.1 (for $n \geq 3$) and Theorem 1.3, we have the following theorem:

Theorem 3.1. For any integer $n \geq 2$, $rn((P_n^{+++})_2) = \begin{cases} 3 & \text{if } n = 2 \\ 8p^2 + 3 & \text{if } n \equiv 1(mod\, 4) \\ 8p^2 + 8p + 4 & \text{if } n \equiv 2(mod\, 4) \\ 8p^2 + 8p + 5 & \text{if } n \equiv 3(mod\, 4) \\ 8p^2 + 16p + 10 & \text{if } n \equiv 0(mod\, 4) \end{cases}$

where $p = \lfloor \frac{2n-1}{8} \rfloor$

In the next sections of this paper, we prove the following theorems, that correspond to the remaining seven k^{th}-transformation graphs of the path P_n for the case $k = 2$.

Theorem 3.2. For any integer $n \geq 6$, $rn((P_n^{---})_2) = \begin{cases} 16 & \text{if } n = 6 \\ 2n - 1 & \text{if } n \geq 7 \end{cases}$

Theorem 3.3. For any integer $n \geq 2$, $rn((P_n^{--+})_2) = \begin{cases} 4 & \text{if } n = 2 \\ 6 & \text{if } n = 3 \\ 2n - 1 & \text{if } n \geq 4 \end{cases}$

Theorem 3.4. For any integer $n \geq 6$, $rn((P_n^{-+-})_2) = \begin{cases} 18 & \text{if } \ n = 6 \\ 24 & \text{if } \ n = 7 \\ 2n-1 & \text{if } \ n \geq 8 \end{cases}$

Theorem 3.5. For any integer $n \geq 2$, $rn((P_n^{-++})_2) = \begin{cases} 4 & \text{if } \ n = 2 \\ 7 & \text{if } \ n = 3 \\ 8 & \text{if } \ n = 4 \\ 2n-1 & \text{if } \ n = 5, 6 \\ 3n+3 & \text{if } \ 7 \leq n \leq 9 \\ 3n+2 & \text{if } \ n = 10 \\ 3n+1 & \text{if } \ n = 11 \\ 3n & \text{if } \ n = 12 \\ 3n-1 & \text{if } \ n \geq 13 \end{cases}$

Theorem 3.6. For any integer $n \geq 5$, $rn((P_n^{+--})_2) = \begin{cases} 21 & \text{if } \ n = 5 \\ 20 & \text{if } \ n = 6 \\ 2n-1 & \text{if } \ n \geq 7 \end{cases}$

Theorem 3.7. For any integer $n \geq 2$,

$$rn((P_n^{+-+})_2) = \begin{cases} 3 & \text{if } \ n = 2 \\ 8 & \text{if } \ n = 3 \\ 9 & \text{if } \ n = 4 \\ 10 & \text{if } \ n = 5 \\ 3n+2 & \text{if } \ 6 \leq n \leq 8 \\ 3n+1 & \text{if } \ n = 9 \\ 3n & \text{if } \ n = 10 \\ 3n-1 & \text{if } \ n = 11 \\ 3n-2 & \text{if } \ n \geq 12 \end{cases}$$

Theorem 3.8. For any integer $n \geq 4$, $rn((P_n^{++-})_2) = \begin{cases} 12 & \text{if } \ n = 4 \\ 2n-1 & \text{if } \ n \geq 5 \end{cases}$

Note: For those values of n not indicated in the above theorems, the graph G is disconnected or trivial.

4 Diameter and a lower bound

Since the diameter of the graph P_n^{xyz} is a constant for each $x, y, z \in \{+, -\}$ except for $xyz = +++$, the proof of the following theorem is an easy exercise.

Theorem 4.1. For a given positive integer $n \geq 2$,

$$diam((P_n^{xyz})_2) = \begin{cases} 1 \quad if & [(xyz = + - +) \ \& \ (n = 2)] \\ 2 \quad if & [(xyz = - - -) \ \& \ (n \geq 7)] \\ \qquad or & [(xyz = - - +) \ \& \ (n \geq 2)] \\ \qquad or & [(xyz = - + -) \ \& \ (n \geq 8)] \\ \qquad or & [(xyz = - + +) \ \& \ (2 \leq n \leq 6)] \\ \qquad or & [(xyz = + - -) \ \& \ (n \geq 7)] \\ \qquad or & [(xyz = + - +) \ \& \ (3 \leq n \leq 5)] \\ \qquad or & [(xyz = + + -) \ \& \ (n \geq 5)] \\ 3 \quad if & [(xyz = - - -) \ \& \ (n = 6)] \\ \qquad or & [(xyz = - + -) \ \& \ (n = 6, 7)] \\ \qquad or & [(xyz = - + +) \ \& \ (n \geq 7)] \\ \qquad or & [(xyz = + - -) \ \& \ (n = 6)] \\ \qquad or & [(xyz = + - +) \ \& \ (n \geq 6)] \\ \qquad or & [(xyz = + + -) \ \& \ (n = 4)] \\ 4 \quad if & [(xyz = + - -) \ \& \ (n = 5)] \end{cases}$$

Let f be a radio labeling of the graph G of order n and $x_1, x_2, \ldots, x_n$ be the re-labeling of the vertices of G such that $f(x_i) < f(x_j)$ whenever $i < j$. The sequence $x_1, x_2, \ldots, x_n$ is called an f-sequence of G. Further for an f-sequence of G, we see that

$$\begin{aligned} f(x_n) - f(x_1) = \sum_{i=1}^{n-1} [f(x_{i+1}) - f(x_i)] &\geq \sum_{i=1}^{n-1} [diam(G) + 1 - d(x_i, x_{i+1})] \\ &= \sum_{i=1}^{n-1} [diam(G) + 1] - \sum_{i=1}^{n-1} d(x_i, x_{i+1}) \end{aligned}$$

Therefore,

$$f(x_n) \geq f(x_1) + (|V(G)| - 1)(diam(G) + 1) - S \tag{2}$$

where $S = \sum_{i=1}^{|V(G)|-1} d(x_i, x_{i+1})$.

In view of inequality (2), we see that $f(x_n)$ is minimum, for any radio labeling f, if $f(x_1) = 1$ and S is as maximum as possible. But, taking $d(x_i, x_{i+1}) = k_i$, we see that

$$S = \sum_{i=1}^{|V(G)|-1} k_i = \sum_{i=1}^{diam(G)} i\alpha_i \tag{3}$$

with $\sum_{i=1}^{|V(G)|-1} \alpha_i = 2n-2$, $\alpha_i \in Z^+ \cup \{0\}$, and $\alpha_i \leq \min\{|V(\langle P_G(i)\rangle)|-1, |P_G(i)|\}$. Thus, finding the radio number of G is equivalent to solving the Linear Integer Programming Problem (3) in G such that the vertex pairs corresponding to a feasible solution set form a Hamiltonian path in $K(G)$.

Lemma 4.2. If f be a radio labeling of a graph G of diameter d with equality in (2), then for any three consecutive vertices u, v, w in the f-sequence of G with $\{u,v\}, \{v,w\} \in P_G(d)$, $d(u,w) \geq d-1$.

Proof: Suppose to the contrary that $d(u,w) \leq d-2$, then, by the definition of radio labeling, it follows that $f(w) = f(u)+2$. Hence, $2 = |f(w)-f(u)| = 1 + diam(G) - d(u,w) \geq 1 + d - (d-2) = 3$, a contradiction. ■

Let $G = P_n^{xyz}$. Then $|V(G)| = 2n-2$. We now find optimal integer solutions for the above L.P.P. (3) in different cases as follows.

Case 1: $xyz = ---$.

When $n = 6$, the diameter of G is 3, $|V(\langle P_G(3)\rangle)| = 6$ and $|P_G(3)| = 6$. So the maximum possible value of $\alpha_3 = 5$, (since $\alpha_i \leq \max\{|V(\langle P_G(i)\rangle)|-1, |P_G(i)|\}$) and the next maximum possible value of $\alpha_2 = 5$ (since $\alpha_3 + \alpha_2 + \alpha_1 = 2n-2 = 10$), the optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{3} i\alpha_i = 25$. Substituting these in inequality (2), gives $rn(G) \geq f(x_n) = 1 + 10(4) - 25 = 16$. Thus, in this case, in view of inequality (1) (for $n \geq 7$), we have;

$$rn((P_n^{---})_2) \geq \begin{cases} 16 & \text{if } n = 6 \\ 2n-1 & \text{if } n \geq 7 \end{cases} \tag{4}$$

Case 2: $xyz = --+$.

When $n = 2$, $G \cong P_3$ and hence the result follows by Theorem 1.2. When $n = 3$, the diameter of G is 2 and $|P_G(2)| = 4$ and maximum possible value of $\alpha_2 = 3$ (since $\langle P_G(2)\rangle \cong C_3 \cup P_2$, and hence maximum possible k_2-pairs in f-sequence of G is 3) and with the next maximum possible value of $\alpha_1 = 1$ (since $\alpha_2 + \alpha_1 = 2n-2 = 4$), the optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{2} i\alpha_i = 7$. Substituting these in inequality (2), we get $rn(G) \geq$

$f(x_n) = 1 + 4(3) - 7 = 6$. Thus, in this case, in view of inequality (1) (for $n \geq 4$), we have;

$$rn((P_n^{--+})_2) \geq \begin{cases} 4 & \text{if} \quad n = 2 \\ 6 & \text{if} \quad n = 3 \\ 2n-1 & \text{if} \quad n \geq 4 \end{cases} \tag{5}$$

Case 3: $xyz = -+-$.

When $n = 6$, the diameter of G is 3 and α_3 = maximum possible value = $|P_G(3)| = 3$, and with the next maximum value of $\alpha_2 = 7$ (since $\alpha_3 + \alpha_2 + \alpha_1 = 2n - 2 = 10$), as above, inequality (2) yields $rn(G) \geq f(x_n) = 1 + (10)(4) - (23) = 18$.

When $n = 7$, the diameter of G is 3 and α_3 = maximum possible value = $|P_G(3)| = 1$, and with the next maximum value of $\alpha_2 = 11$ (since $\alpha_3 + \alpha_2 + \alpha_1 = 2n - 2 = 12$), inequality (2) gives $rn(G) \geq f(x_n) = 1 + (12)(4) - (25) = 24$.

Thus, in this case, in view of inequality (1) (for $n \geq 8$), we have

$$rn((P_n^{-+-})_2) \geq \begin{cases} 18 & \text{if} \quad n = 6 \\ 24 & \text{if} \quad n = 7 \\ 2n-1 & \text{if} \quad n \geq 8 \end{cases} \tag{6}$$

Case 4: $xyz = -++$.

When $n = 2$, the diameter of G is 2 and $|P_G(2)| = 1$. The maximum possible value of $\alpha_2 = 1$ and with the next maximum possible value of $\alpha_1 = 1$, inequality (2) gives $rn(G) \geq f(x_n) = 1 + (3)(2) - (3) = 4$.

When $n = 3$, the diameter of G is 2 and $|P_G(2)| = 3$. The maximum possible value of $\alpha_2 = 2$ (since $\langle P_G(2) \rangle \cong C_3$) and with the next maximum possible value of $\alpha_1 = 2$ (since $\alpha_2 + \alpha_1 = 2n - 2 = 4$) inequality (2) gives $rn(G) \geq f(x_n) = 1 + (4)(3) - (6) = 7$.

When $n = 4$, the diameter of G is 2 and $|V(\langle P_G(2) \rangle)| = 6$. Thus, the maximum possible value of $\alpha_2 = 5$, and maximum possible value of $\alpha_1 = 1$ (since $\alpha_2 + \alpha_1 = 2n - 2 = 6$). So inequality (2), gives $rn(G) \geq f(x_n) = 1 + (6)(3) - (11) = 8$.

When $n = 5, 6$, result follows by inequality 1 as $rn(G) \geq |V(G)| = 2n - 1$.

When $7 \leq n \leq 9$, the diameter of G is 3. In view of Lemma 4.2 and Figure 2, it is easy to observe that at most one k_3 edge is incident with each vertex in any f-sequence of G. Therefore, at most 1, 2, and 3, k_3-pairs lie in any f-sequence of G respectively when

$n = 7, 8$, and 9. So, $\alpha_3 = n - 6$ for $n \in \{7, 8, 9\}$. The next maximum value for $\alpha_2 = n + 4$ (since $\alpha_3 + \alpha_2 + \alpha_1 = 2n - 2$), and hence by inequality (2), we get $rn(G) \geq f(x_n) = 1 + (2n - 2)(4) - (5n - 10) = 3n + 3$.

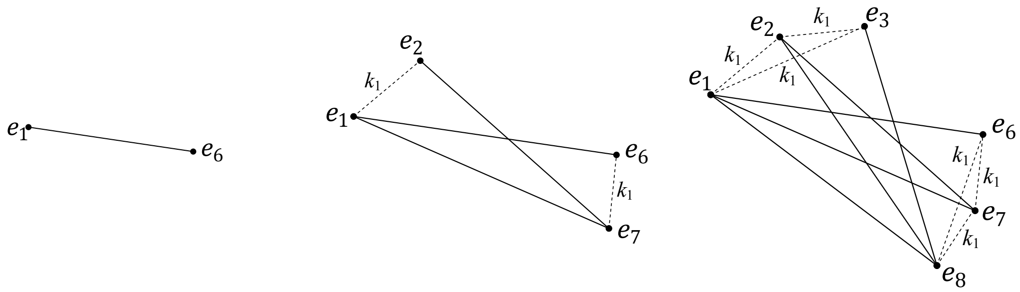

Figure 2: The graphs $P_G(3)$ for the cases $n = 7, 8$ and 9.

When $n = 10$, the diameter of G is 3. In view of Lemma 4.2 and Figure 3, it is easy to observe that each f-sequence of G has at most one k_3 edge incident with the vertices in $\langle P_G(3) \rangle$ except e_1 and e_8. So α_3 = number of k_3 edges incident with the vertices in any f sequence H of G = $\frac{1}{2} \sum_{xy \in P_G(3)} \deg_H(x) = \frac{1}{2}[2 + 2 + 1 + 1 + 1 + 1 + 1 + 1] = 5$ and the next maximum value for $\alpha_2 = 13$ (since $\alpha_3 + \alpha_2 + \alpha_1 = 2n - 2 = 18$). From the optimal value of L.P.P. (3), $\max S = \sum_{i=1}^{3} i\alpha_i = 41$ and hence by inequality (2), we get $rn(G) \geq f(x_n) = 1 + (2n - 2)(4) - (41) = 32 = 3n + 2$ when $n = 10$.

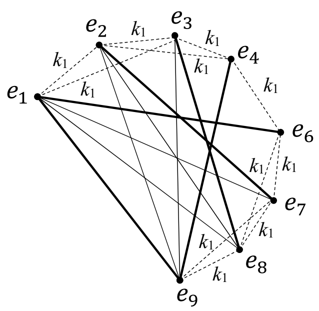

Figure 3: The graph $\langle P_G(3) \rangle$, for $n = 10$.

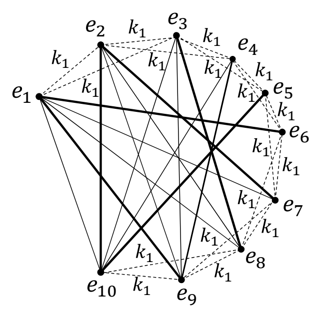

Figure 4: The graph $\langle P_G(3) \rangle$, for $n = 11$.

When $n = 11$, the diameter of G is 3. In view of Lemma 4.2 and Figure 4, it is easy to observe that each f-sequence H of G has at most one k_3 edge incident with e_i in $\langle P_G(3) \rangle$ for every $3 \leq i \leq 8$. So α_3 = $\frac{1}{2} \sum_{xy \in P_G(3)} \deg_H(x) = \frac{1}{2}[2+2+2+2+1+1+1+1+1+1] = 7$ and the next maximum value for $\alpha_2 = 13$ (since $\alpha_3 + \alpha_2 + \alpha_1 = 2n - 2 = 20$). From the optimal value of L.P.P. (3), $\max S = \sum_{i=1}^{3} i\alpha_i = 47$ and hence by inequality (2), we get $rn(G) \geq f(x_n) = 1 + (2n - 2)(4) - (47) = 34 = 3n + 1$ when $n = 11$.

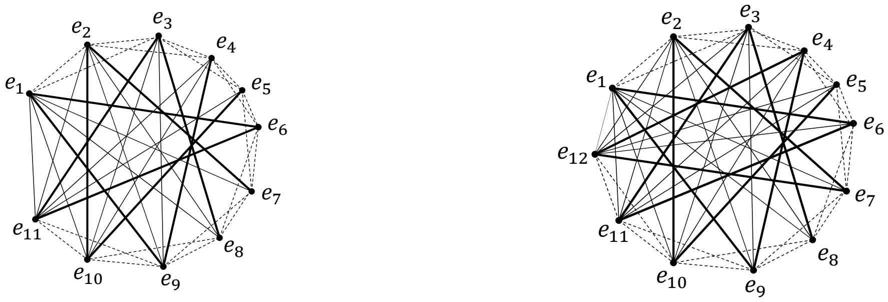

Figure 5: The graph $\langle P_G(3)\rangle$ for the case $n = 12, 13$.

When $n = 12$, the diameter of G is 3. In view of Lemma 4.2 and Figure 5, it is easy to observe that each f-sequence H of G has at most one k_3 edge incident with e_i in $\langle P_G(3)\rangle$ for $i = 4, 5, 7, 8$. So $\alpha_3 = \frac{1}{2}\sum_{xy\in P_G(3)} \deg_H(x) = \frac{1}{2}[2+2+2+2+2+2+2+2+1+1+1+1] = 9$ and the next maximum value for $\alpha_2 = 13$ (since $\alpha_3 + \alpha_2 + \alpha_1 = 2n - 2 = 22$). From the optimal value of L.P.P. (3), $\max S = \sum_{i=1}^{3} i\alpha_i = 53$ and hence by inequality (2), we get $rn(G) \geq f(x_n) = 1 + (2n - 2)(4) - (53) = 36 = 3n$ when $n = 12$.

When $n \geq 13$, the diameter of G is 3, $|V(\langle P_G(i)\rangle)| = n - 1$ and $|P_G(i)| = \frac{(n-4)(n-5)}{2}$. Hence maximum value for $\alpha_3 = \min\{n-2, \frac{(n-4)(n-5)}{2}\} = n-2$ and the next maximum value for $\alpha_2 = n$. Therefore, the optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{3} i\alpha_i = 5n - 6$ and hence by inequality (2), $rn(G) \geq f(x_n) = 1 + (2n - 2)(4) - (5n - 6) = 3n - 1$.

Thus, we have;

$$rn((P_n^{-++})_2) \geq \begin{cases} 4 & \text{if} \quad n = 2 \\ 7 & \text{if} \quad n = 3 \\ 8 & \text{if} \quad n = 4 \\ 2n-1 & \text{if} \quad n = 5, 6 \\ 3n+3 & \text{if} \quad 7 \leq n \leq 9 \\ 3n+2 & \text{if} \quad n = 10 \\ 3n+1 & \text{if} \quad n = 11 \\ 3n & \text{if} \quad n = 12 \\ 3n-1 & \text{if} \quad n \geq 13 \end{cases} \tag{7}$$

Case 5: $xyz = +--$.

When $n = 5$, the diameter of G is 4 and $\langle P_G(4) \cup P_G(3)\rangle$ which is isomorphic to the bistar $B_{3,3}$ as shown in Figure 6. Therefore, the two feasible choices are; $\alpha_4 = 1, \alpha_3 = 2, \alpha_2 = 5$ or $\alpha_4 = 0, \alpha_3 = 4, \alpha_2 = 4$. In either of the cases we get $\max S = 20$. Substituting these in inequality (2), gives $rn(G) \geq f(x_n) = 1 + (8)(5) - (20) = 21$.

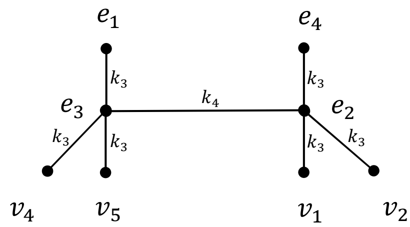

Figure 6: The graph $\langle P_G(4) \cup P_G(3)\rangle$ for the case $n = 5$.

When $n = 6$, the diameter of G is 3 and α_3 = maximum possible value =$|P_G(3)| = 1$, and the next maximum possible value of $\alpha_2 = 9$, the optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{3} i\alpha_i = 21$. Substituting these in inequality (2), gives $rn(G) \geq f(x_n) = 1 + (10)(4) - (21) = 20$.

Thus, in this case, in view of inequality (1) (for $n \geq 7$), we have;

$$rn((P_n^{+--})_2) \geq \begin{cases} 21 & \text{if} \quad n = 5 \\ 20 & \text{if} \quad n = 6 \\ 2n-1 & \text{if} \quad n \geq 7 \end{cases} \qquad (8)$$

Case 6: $xyz = + - +$.

When $n = 2$, the result follows from Theorem 1.3. When $n = 3$, the diameter of G is 2 and α_2 = maximum possible value =$|P_G(2)| = 1$ and the next maximum value for $\alpha_1 = 3$. Substituting these in inequality (2), we get $rn(G) \geq f(x_n) = 1 + (4)(3) - 5 = 8$. When $n = 4$, the diameter of G is 2 and $|P_G(2)| = 6$, $|V(\langle P_G(2)\rangle)| = 5$. Thus maximum possible value of α_2 =4, and the next maximum value for $\alpha_1 = 2$. The optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{2} i\alpha_i = 10$. Substituting these in inequality (2), gives $rn(G) \geq f(x_n) = 1 + (6)(3) - (10) = 9$. When $n = 5$, the diameter of G is 2 and and $|P_G(2)| = 14$, $|V(\langle P_G(2)\rangle)| = 8$. Thus maximum possible value of α_2 =7 and the next maximum value for $\alpha_1 = 1$. The optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{2} i\alpha_i = 15$. Substituting these in inequality (2), gives $rn(G) \geq f(x_n) = 1 + (8)(3) - (15) = 10$.

When $6 \leq n \leq 8$, the diameter of G is 3. In view of Lemma 4.2 and Figure 7, it is easy to observe that at most one k_3 edge incident with each vertex in any f-sequence of G. Therefore, at most $1, 2$ and 3, k_3-pairs lie in any f-sequence of G respectively when $n = 6, 7$ and 8. So, $\alpha_3 = n - 5$ for $n \in \{6, 7, 8\}$. The next maximum value for $\alpha_2 = n + 3$ (since $\alpha_3 + \alpha_2 + \alpha_1 = 2n - 2$) and hence by inequality (2), we get $rn(G) \geq f(x_n) = 1 + (2n - 2)(4) - (5n - 9) = 3n + 2$.

When $n = 9$, the diameter of G is 3. In view of Lemma 4.2 and Figure 8, it is easy to observe

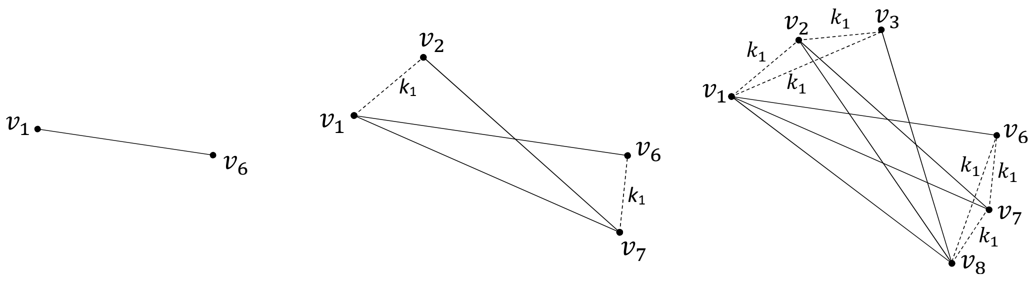

Figure 7: The graphs $P_G(3)$ for the case $n = 6, 7$ and 8.

that each f-sequence of G has at most one k_3 edge incident with the vertices in $\langle P_G(3)\rangle$ except v_1 and v_8. So α_3 = number of k_3 edges incident with the vertices in any f sequence H of G = $\frac{1}{2}\sum_{xy\in P_G(3)} \deg_H(x) = \frac{1}{2}[2{+}2{+}1{+}1{+}1{+}1{+}1{+}1] = 5$ and the next maximum value for $\alpha_2 = 11$ (since $\alpha_3{+}\alpha_2{+}\alpha_1 = 2n{-}2$). From the optimal value of L.P.P. (3), $\max S = \sum_{i=1}^{3} i\alpha_i = 37$ and hence by inequality (2), we get $rn(G) \geq f(x_n) = 1 + (2n-2)(4) - (37) = 28 = 3n+1$ when $n = 9$.

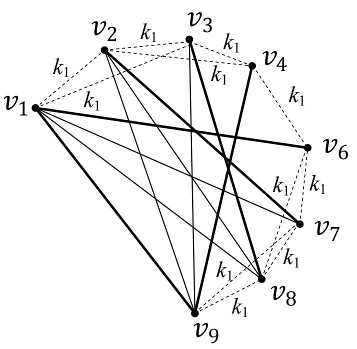

Figure 8: The graph $P_G(3)$ for the case $n = 9$.

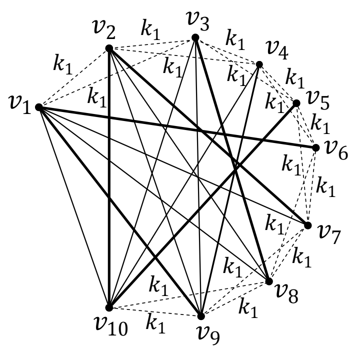

Figure 9: The graph $P_G(3)$ for the case $n = 10$.

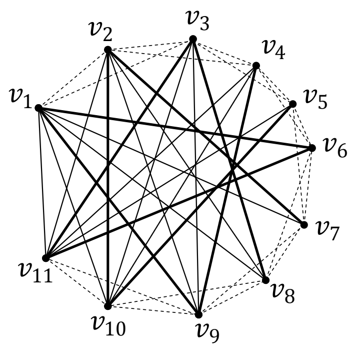

Figure 10: The graph $P_G(3)$ for the case $n = 11$.

When $n = 10$, the diameter of G is 3. In view of Lemma 4.2 and Figure 9, it is easy to observe that each f-sequence H of G has at most one k_3 edge incident with v_i in $\langle P_G(3)\rangle$ for every $3 \leq i \leq 8$. So $\alpha_3 = \frac{1}{2}\sum_{xy\in P_G(3)} \deg_H(x) = \frac{1}{2}[2{+}2{+}2{+}2{+}1{+}1{+}1{+}1{+}1{+}1] = 7$ and the next maximum value for $\alpha_2 = 11$. Thus, the optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{3} i\alpha_i = 43$. Substituting this in inequality (2) gives $rn(G) \geq f(x_n) = 1{+}(2n{-}2)(4){-}(43) = 30 = 3n$ when $n = 10$.

When $n = 11$, the diameter of G is 3. In view of Lemma 4.2 and Figure 10, it is easy to observe that each f-sequence H of G has at most one k_3 edge incident with v_i in $\langle P_G(3)\rangle$ for every $4 \leq i \leq 8$. So $\alpha_3 = \frac{1}{2}\sum_{xy\in P_G(3)} \deg_H(x) = \frac{1}{2}[2+2+2+2+2+2+2+1+1+1+1] = 9$

and the next maximum value for $\alpha_2 = 11$.Thus, the optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{3} i\alpha_i = 49$ and hence by inequality (2), we get $rn(G) \geq f(x_n) = 1+(2n-2)(4)-(49) = 32 = 3n-1$ when $n = 11$.

When $n \geq 12$, the diameter of G is 3. $|V(\langle P_G(i)\rangle)| = n$ and $|P_G(i)| = \frac{(n-4)(n-5)}{2}$. Hence maximum value for $\alpha_3 = \min\{n-1, \frac{(n-4)(n-5)}{2}\} = n-1$ and the next maximum value for $\alpha_2 = n-1$. Thus, the optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{3} i\alpha_i = 5n-5$ and hence by inequality (2), we get $rn(G) \geq f(x_n) = 1 + (2n-2)(4) - (5n-5) = 3n-2$.

Thus, we have

$$rn((P_n^{+-+})_2) \geq \begin{cases} 3 & \text{if } n = 2 \\ 8 & \text{if } n = 3 \\ 9 & \text{if } n = 4 \\ 10 & \text{if } n = 5 \\ 3n+2 & \text{if } 6 \leq n \leq 8 \\ 3n+1 & \text{if } n = 9 \\ 3n & \text{if } n = 10 \\ 3n-1 & \text{if } n = 11 \\ 3n-2 & \text{if } n \geq 12 \end{cases} \tag{9}$$

Case 7: $xyz = ++-$.

When $n = 4$, the diameter of G is 3 and $\langle P_G(3)\rangle \cong P_3$ where the end vertices are a k_1 pair. Therefore, by Lemma 4.2, at most one pair of adjacent vertices of $\langle P_G(3)\rangle$ lies in every f-sequence of G. Hence, $\alpha_3 = 1$ and so $\alpha_2 = 5$ (since $\alpha_3 + \alpha_2 + \alpha_1 = 2n - 2 = 6$), Thus, the optimal value of L.P.P. (3) is $\max S = \sum_{i=1}^{3} i\alpha_i = 13$. Substituting these in inequality (2), gives $rn(G) \geq f(x_n) = 1 + (6)(4) - (13) = 12$. Thus, in this case, in view of inequality (1) (for $n \geq 5$), we have;

$$rn((P_n^{++-})_2) \geq \begin{cases} 12 & \text{if } n = 4 \\ 2n-1 & \text{if } n \geq 5 \end{cases} \tag{10}$$

5 Upper bound and a Radio labeling

We begin with the following lemma:

Lemma 5.1. If $x_1, x_2, \ldots, x_n$ is a sequence of vertices of $G = (P_n^{xyz})_2$ where $xyz = +-+$ or $-++$ such that

$$d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) \leq 1 + diam(G) + d(x_i, x_{i+2}) \text{ for every } i, 1 \leq i \leq 2n-3,$$

then the function $f : V(G) \to Z^+$ defined by

$$f(x_{i+1}) = f(x_i) + 1 + diam(G) - d(x_i, x_{i+1}) \tag{11}$$

is a radio labeling of G.

Proof: Consider the vertices x_i, x_j. Without loss of generality we take $j > i$.

Case 1: $j = i + 1$.

In this case, it follows from the definition of f, in equation (11), that $f(x_j) - f(x_i) \geq 1 + diam(G) - d(x_j, x_i)$.

Case 2: $j = i + 2$.

In this case, $|f(x_j) - f(x_i)| = f(x_{i+2}) - f(x_{i+1}) + f(x_{i+1}) - f(x_i) = 1 + diam(G) - d(x_{i+2}, x_{i+1}) + 1 + diam(G) - d(x_{i+1}, x_i) = 2(1 + diam(G)) - [d(x_{i+2}, x_{i+1}) + d(x_{i+1}, x_i)] \geq 2(1 + diam(G)) - [(1 + dim(G)) + d(x_{i+2}, x_i)] = 1 + diam(G) - d(x_{i+2}, x_i) = 1 + diam(G) - d(x_j, x_i)$

Case 3: $j \geq i + 3$.

In this case, as f is an injective function, it follows that $|f(x_j) - f(x_i)| \geq 3$ and hence $f(x_j) - f(x_i) + d(x_i, x_j) \geq 3 + 1 = 1 + 3 \geq 1 + diam(G)$ (since $diam(G) \leq 3$ by Theorem 4.1).

Thus f is a radio labeling. ∎

Lemma 5.2. $rn((P_n^{xyz})_2) \leq 2n - 1$ if $\begin{cases} \quad xyz = - - - & \text{and} \quad n \geq 7 \\ or \quad xyz = - - + & \text{and} \quad n \geq 4 \\ or \quad xyz = - + - & \text{and} \quad n \geq 8 \\ or \quad xyz = + - - & \text{and} \quad n \geq 7 \\ or \quad xyz = + + - & \text{and} \quad n \geq 5 \end{cases}$

Proof: Let $G = (P_n^{xyz})_2$. Let $v_1, v_2, \ldots v_n \in V(P_n)$ and $e_1, e_2, \ldots e_{n-1} \in E(P_n)$. Let $x_i x_j \in E(G)$. Since $diam(G) = 2$ (by Theorem 4.1), to prove that an injective function f is a radio labeling of G, it suffices to show that $|f(x_i) - f(x_j)| \geq 2$ whenever $d(x_i, x_j) = 1$ for all possible values of i, j, or equivalently $|f(x_i) - f(x_j)| = 1$ implies $x_i x_j \notin E(G)$.

Case a: When $xyz = - - -$ or $- + -$ or $+ - -$ or $+ + -$.

Define $f : V(G) \to Z^+$ such that $f(v_i) = 2i - 1$ for $1 \leq i \leq n$ and $f(e_i) = 2i$ for $1 \leq i \leq n - 1$. Then f is an injection. Further, if $|f(x_j) - f(x_i)| = 1$, then either $x_i = v_k$ and $x_j = e_k$, or, $x_i = e_k$ and $x_j = v_{k+1}$, for some $1 \leq k \leq n - 1$. In either of the cases, x_i is incident with x_j in $P - n$ and hence $x_i x_j \notin E(P_n^{xyz})$ (since $z = -1$ in this case). So f is a radio labeling. Therefore $rn(G) \leq spanf = f(v_n) = 2(n) - 1 = 2n - 1$.

Case b: When $xyz = --+$.

Define $f : V(G) \to Z^+$ such that $f(v_i) = i$, $f(e_i) = n + i$ for $1 \leq i \leq n$. Then f is an injection. Further, if $f(x_j) > f(x_i)$ and $f(x_j) - f(x_i) = 1$, then either $x_i = v_k$ and $x_j = v_{k+1}$, or, $x_i = e_k$ and $x_j = e_{k+1}$, or $x_i = v_n$ and $x_j = e_1$, for some $1 \leq k \leq n - 1$. In the first two cases x_i is adjacent to x_j in P_n and in the last case they are not incident to each other. Hence $x_i x_j \notin E(P_n^{xyz})$ (respectively as $x = -1$, $y = -1$ and $z = +$ in this case). So f is a radio labeling. Therefore $rn(G) \leq spanf = f(v_n) = 2(n) - 1 = 2n - 1$. ∎

We now execute a minimal radio labeling for each of the cases and get the required upper bound.

Case 1: $xyz = ---$.

When $n = 6$, from the radio labeling given in Figure 11, it follows that

$$rn((P_6^{---})_2) \leq 16 \tag{12}$$

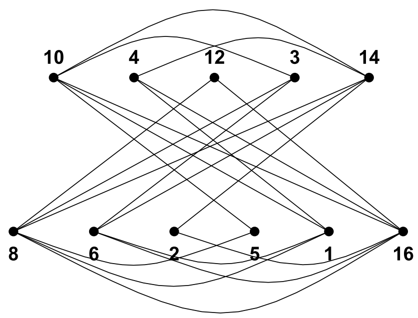

Figure 11: A radio labeling of $(P_6^{---})_2$

Now, inequality (4), inequality (12) and Lemma 5.2 together prove Theorem 3.2.

Case 2: $xyz = --+$.

When $n = 2, 3$, it follows from Figure 12 and Figure 13 that

$$rn((P_n^{--+})_2) \leq \begin{cases} 4 & \text{if} \quad n = 2 \\ 6 & \text{if} \quad n = 3 \end{cases} \tag{13}$$

Now, inequality (5), inequality (13) and Lemma 5.2 together prove Theorem 3.3.

Case 3: $xyz = -+-$.

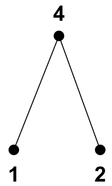

Figure 12: A radio labeling of $(P_2^{--+})_2$

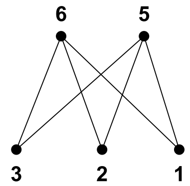

Figure 13: A radio labeling of $(P_3^{--+})_2$

When $n = 6, 7$, it follows from Figure 5 that

$$rn((P_n^{-+-})_2) \leq \begin{cases} 18 & \text{if} \quad n = 6 \\ 24 & \text{if} \quad n = 7 \end{cases} \tag{14}$$

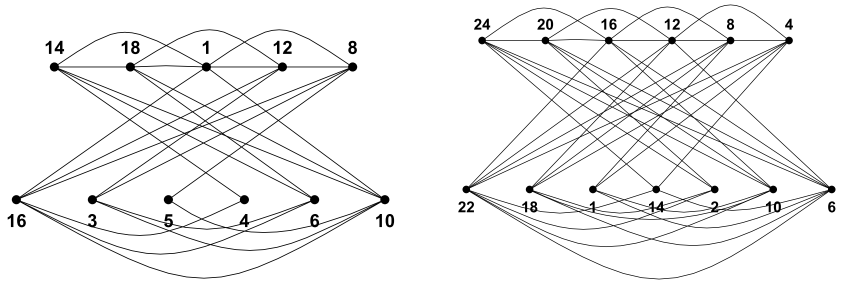

Figure 14: A radio labeling of $(P_6^{-+-})_2$ and $(P_7^{-+-})_2$

Now, inequality (6), inequality (14) and Lemma 5.2 together prove Theorem 3.4.

Case 4: $xyz = -++$.

When $n = 2, 3, 4$, it follows from Figure 15 that

$$rn((P_n^{-++})_2) \leq \begin{cases} 4 & \text{if} \quad n = 2 \\ 7 & \text{if} \quad n = 3 \\ 8 & \text{if} \quad n = 4 \end{cases} \tag{15}$$

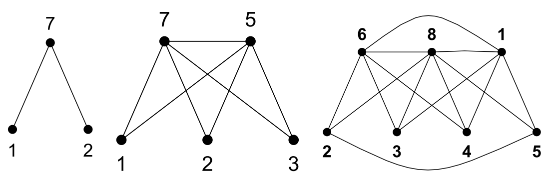

Figure 15: A radio labeling of $(P_2^{-++})_2$, $(P_3^{-++})_2$ and $(P_4^{-++})_2$

When $5 \leq n \leq 13$, we give below a sequence $x_1, x_2, \ldots, x_{2n-1}$ of vertices of G to label in such a way that $f(x_i) = f(x_{i-1}) + 1 + diam(G) - d(x_i, x_{i+1})$ with $f(x_1) = 1$. It is easy to verify that f is a radio labeling (with the difference of labels of two consecutive vertices in the sequence indicated above the arrow) as $1+diam(G)-d(x_i, x_{i+1}) \leq 2$ for each $1 \leq i \leq 2n-2$.

For $n = 5$.

$x_1 = e_3 \xrightarrow{+1} v_1 \xrightarrow{+1} e_4 \xrightarrow{+1} v_2 \xrightarrow{+1} v_3 \xrightarrow{+1} v_4 \xrightarrow{+1} e_1 \xrightarrow{+1} v_5 \xrightarrow{+1} e_2 = x_{2n-1}$.

From this sequence, we get

$$\begin{aligned} rn((P_5^{-++})_2) \leq spanf = f(x_{2n-1}) &= f(x_1) + \sum_{k=2}^{k=2n-1} [f(x_i) - f(x_{i-1})] \\ &= 1 + 8(1) = 9. \end{aligned} \tag{16}$$

For $n = 6$.

$x_1 = f(e_5) \xrightarrow{+1} v_3 \xrightarrow{+1} v_2 \xrightarrow{+1} e_4 \xrightarrow{+1} v_1 \xrightarrow{+1} e_3 \xrightarrow{+1} v_6 \xrightarrow{+1} e_2 \xrightarrow{+1} v_5 \xrightarrow{+1} e_1 \xrightarrow{+1} v_4 = x_{11}$.

From this sequence, we get

$$\begin{aligned} rn((P_6^{-++})_2) \leq spanf = f(x_{11}) &= f(x_1) + \sum_{k=2}^{11} [f(x_i) - f(x_{i-1})] \\ &= 1 + 10(1) = 11. \end{aligned} \tag{17}$$

For $n = 7$.

$x_1 = e_1 \xrightarrow{+1} e_6 \xrightarrow{+2} e_2 \xrightarrow{+2} e_5 \xrightarrow{+2} v_1 \xrightarrow{+2} e_3 \xrightarrow{+2} v_7 \xrightarrow{+2} e_4 \xrightarrow{+2} v_2 \xrightarrow{+2} v_3 \xrightarrow{+2} v_4 \xrightarrow{+2} v_5 \xrightarrow{+2} v_6 = x_{13}$.

From this sequence, we get

$$\begin{aligned} rn((P_7^{-++})_2) \leq spanf = f(x_{13}) &= f(x_1) + \sum_{k=2}^{13} [f(x_i) - f(x_{i-1})] \\ &= 1 + 1(1) + 11(2) = 24. \end{aligned} \tag{18}$$

For $n = 8$.

$x_1 = e_1 \xrightarrow{+1} e_6 \xrightarrow{+2} e_2 \xrightarrow{+1} e_7 \xrightarrow{+2} e_3 \xrightarrow{+2} v_1 \xrightarrow{+2} e_4 \xrightarrow{+2} v_2 \xrightarrow{+2} e_5 \xrightarrow{+2} v_3 \xrightarrow{+2} v_4 \xrightarrow{+2} v_5 \xrightarrow{+2} v_6 \xrightarrow{+2} v_7 \xrightarrow{+2} v_8 = x_{15}$.

From this sequence, we get

$$\begin{aligned} rn((P_8^{-++})_2) \leq spanf = f(x_{15}) &= f(x_1) + \sum_{k=2}^{15} [f(x_i) - f(x_{i-1})] \\ &= 1 + 2(1) + 12(2) = 27. \end{aligned} \tag{19}$$

For $n = 9$.

$x_1 = e_1 \xrightarrow{+1} e_6 \xrightarrow{+2} e_2 \xrightarrow{+1} e_7 \xrightarrow{+2} e_3 \xrightarrow{+1} e_8 \xrightarrow{+2} e_4 \xrightarrow{+2} v_1 \xrightarrow{+2} v_2 \xrightarrow{+2} v_3 \xrightarrow{+2} v_4 \xrightarrow{+2} v_5 \xrightarrow{+2} v_6 \xrightarrow{+2} v_7 \xrightarrow{+2} v_8 \xrightarrow{+2} v_9 \xrightarrow{+2} e_5 = x_{17}$.

From this sequence, we get

$$rn((P_9^{-++})_2) \leq span f = f(x_{17}) = f(x_1) + \sum_{k=2}^{17}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 3(1) + 13(2) = 30. \tag{20}$$

For $n = 10$.

$x_1 = e_4 \xrightarrow{+1} e_9 \xrightarrow{+1} e_1 \xrightarrow{+1} e_6 \xrightarrow{+2} e_2 \xrightarrow{+1} e_7 \xrightarrow{+2} e_3 \xrightarrow{+1} e_8 \xrightarrow{+2} e_5 \xrightarrow{+2} v_1 \xrightarrow{+2} v_2 \xrightarrow{+2} v_3 \xrightarrow{+2} v_4 \xrightarrow{+2} v_5 \xrightarrow{+2} v_6 \xrightarrow{+2} v_7 \xrightarrow{+2} v_8 \xrightarrow{+2} v_9 \xrightarrow{+2} v_{10} = x_{19}$.

From this sequence, we get

$$rn((P_{10}^{-++})_2) \leq span f = f(x_{19}) = f(x_1) + \sum_{k=2}^{19}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 5(1) + 12(2) = 29. \tag{21}$$

For $n = 11$.

$e_1 = e_4 = 1 \xrightarrow{+1} e_9 \xrightarrow{+1} e_1 \xrightarrow{+1} e_6 \xrightarrow{+2} v_1 \xrightarrow{+2} e_5 \xrightarrow{+1} e_{10} \xrightarrow{+1} e_2 \xrightarrow{+1} e_7 \xrightarrow{+2} e_3 \xrightarrow{+1} e_8 \xrightarrow{+2} v_2 \xrightarrow{+2} v_3 \xrightarrow{+2} v_4 \xrightarrow{+2} v_5 \xrightarrow{+2} v_6 \xrightarrow{+2} v_) \xrightarrow{+2} v_8 \xrightarrow{+2} v_9 \xrightarrow{+2} v_{10} \xrightarrow{+2} v_{11} = x_{21}$.

From this sequence, we get

$$rn((P_{10}^{-++})_2) \leq span f = f(x_{21}) = f(x_1) + \sum_{k=2}^{21}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 7(1) + 13(2) = 34. \tag{22}$$

For $n = 12$.

$x_1 = e_4 \xrightarrow{+1} e_9 \xrightarrow{+1} e_1 \xrightarrow{+1} e_6 \xrightarrow{+1} e_{11} \xrightarrow{+1} e_3 \xrightarrow{+1} e_8 \xrightarrow{+2} v_1 \xrightarrow{+2} e_5 \xrightarrow{+1} e_{10} \xrightarrow{+1} e_2 \xrightarrow{+1} e_7 \xrightarrow{+2} v_2 \xrightarrow{+2} v_3 \xrightarrow{+2} v_4 \xrightarrow{+2} v_5 \xrightarrow{+2} v_6 \xrightarrow{+2} v_7 \xrightarrow{+2} v_8 \xrightarrow{+2} v_9 \xrightarrow{+2} v_{10} \xrightarrow{+2} v_{11} \xrightarrow{+2} v_{12} = x_{23}$.

From this sequence, we get

$$rn((P_{12}^{-++})_2) \leq span f = f(x_{23}) = f(x_1) + \sum_{k=2}^{23}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 9(1) + 13(2) = 36. \tag{23}$$

.

For $n = 13$.

$x_1 = e_5 \xrightarrow{+1} e_{10} \xrightarrow{+1} e_2 \xrightarrow{+1} e_7 \xrightarrow{+1} e_{12} \xrightarrow{+1} e_4 \xrightarrow{+1} e_9 \xrightarrow{+1} e_1 \xrightarrow{+1} e_6 \xrightarrow{+1} e_{11} \xrightarrow{+1} e_3 \xrightarrow{+1} e_8 \xrightarrow{+2} v_1 \xrightarrow{+2} v_2 \xrightarrow{+2} v_3 \xrightarrow{+2} v_4 \xrightarrow{+2} v_5 \xrightarrow{+2} v_6 \xrightarrow{+2} v_7 \xrightarrow{+2} v_8 \xrightarrow{+2} v_9 \xrightarrow{+2} v_{10} \xrightarrow{+2} v_{11} \xrightarrow{+2} v_{12} \xrightarrow{+2} v_{13} = x_{25}$.

From this sequence, we get

$$\begin{aligned} rn((P_{13}^{-++})_2) \leq spanf = f(x_{25}) &= f(x_1) + \sum_{k=2}^{25}[f(x_i) - f(x_{i-1})] \\ &= 1 + 11(1) + 13(2) = 38. \end{aligned} \tag{24}$$

For $n \geq 14$, Define $f : V(G) \to Z^+$ as

1. $f(e_1) = \lceil \frac{n+1}{5} \rceil$, $f(e_2) = \lfloor 2\left(\frac{n-2}{5}\right) \rfloor + 2$, $f(e_3) = \lfloor 3\left(\frac{n-3}{5}\right) \rfloor + 3$, $f(e_4) = 1$, $f(e_5) = n - \lfloor \frac{n-1}{5} \rfloor$

2. $f(v_1) = \max\{f(e_i) : 1 \leq i \leq n-1\} + 2 = n + 1$

3. $f(e_i) = f(e_{i-4}) + 1$ for $6 \leq i \leq n-1$

4. $f(v_{i+1}) = f(v_i) + 2$ for $1 \leq i \leq n-1$

The function f defined above uses the following sequence of vertices to label the graph.

Clearly f is a radio labeling by Lemma (5.1). In fact, $|f(e_i) - f(e_j)| \geq 1$ if $j \geq i \pm 5$, (that is $d(e_i, e_j) = 3$), $|f(e_i) - f(e_j)| \geq 2$ if $j \geq i \pm 3$, (that is $d(e_i, e_j) = 2$), $|f(e_i) - f(e_j)| \geq 3$ if $j \geq i \pm 1$ (that is $d(e_i, e_j) = 1$). $|f(v_i) - f(e_j)| \geq n + 2$ (since $d(e_i, v_j) = 2, \forall i, j$). Now $|f(v_i) - f(v_j)| \geq 2$ if $j = i \pm 1, 2$ (that is $d(v_i, v_j) = 2$). $|f(v_i) - f(v_j)| \geq 4$ if $j \geq i \pm 3$ (that is $d(e_i, e_j) = 1$). Hence, for all $n \geq 14$;

$$\begin{aligned} rn((P_n^{-++})_2) \leq spanf = f(v_n) &= f(v_{n-1}) + 2 \\ &= [f(v_{n-2}) + 2] + 2 \\ &\vdots \\ &= f(v_1) + 2(n-1) \\ &= n + 1 + 2(n-1) = 3n - 1 \end{aligned} \tag{25}$$

Now, inequalities (15)-(25) and inequality (7) together prove Theorem 3.5.

Case 5: $xyz = +--$.
For $n = 5, 6$, it follows, from Figure 16, that

$$rn((P_n^{+--})_2) \leq \begin{cases} 21, & \text{if} \quad n = 5 \\ 20, & \text{if} \quad n = 6 \end{cases} \tag{26}$$

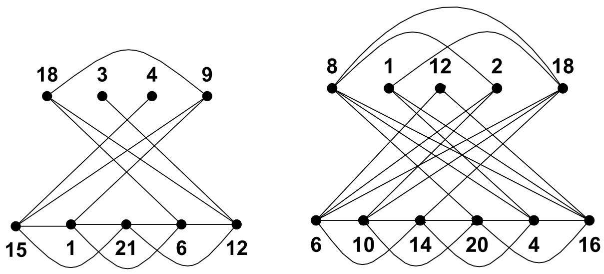

Figure 16: A radio labeling of $(P_5^{+--})_2$ and $(P_6^{+--})_2$

Now, inequality (8), inequality (26) and Lemma 5.2 together prove Theorem 3.6.
Case 6: $xyz = +-+$.
For $n = 3, 4, 5$, it follows, from Figure 17, that

$$rn((P_n^{+-+})_2) \leq \begin{cases} 3 & \text{if} \quad n = 2 \\ 8 & \text{if} \quad n = 3 \\ 9 & \text{if} \quad n = 4 \\ 10 & \text{if} \quad n = 5 \end{cases} \tag{27}$$

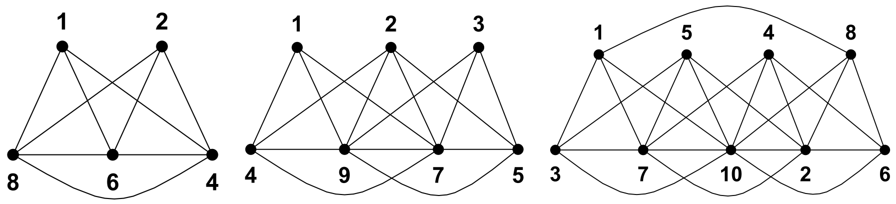

Figure 17: A radio labeling of $(P_3^{+-+})_2$, $(P_4^{+-+})_2$ and $(P_5^{+-+})_2$

When $6 \leq n \leq 12$, we give below a sequence $x_1, x_2, \ldots, x_{2n-1}$ of vertices of G to label in such a way that $f(x_i) = f(x_{i-1}) + 1 + diam(G) - d(x_i, x_{i+1})$ with $f(x_1) = 1$. It is easy to verify that f is a radio labeling (with the difference of labels of two consecutive vertices in the sequence indicated above the arrow) as $1+diam(G)-d(x_i, x_{i+1}) \leq 2$ for each $1 \leq i \leq 2n-2$.

For $n = 6$.
$x_1 = v_1 \xrightarrow{1} v_6 \xrightarrow{+2} v_3 \xrightarrow{+2} e_5 \xrightarrow{+2} e_4 \xrightarrow{+2} v_2 \xrightarrow{+2} v_5 \xrightarrow{+2} e_2 \xrightarrow{+2} e_3 \xrightarrow{+2} e_1 \xrightarrow{+2} v_4 = x_{2-1}$.
From this sequence, we get

$$rn((P_6^{+-+})_2) \leq spanf = f(x_{11}) = f(x_1) + \sum_{k=2}^{11}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 1(1) + 9(2) = 20. \quad (28)$$

For $n = 7$.
$x_1 = v_1 \xrightarrow{+1} v_6 \xrightarrow{+2} v_2 \xrightarrow{+1} v_7 \xrightarrow{+2} v_3 \xrightarrow{+2} e_1 \xrightarrow{+2} v_5 \xrightarrow{+2} e_2 \xrightarrow{+2} e_3 \xrightarrow{+2} e_4 \xrightarrow{+2} e_5 \xrightarrow{+2} e_6 \xrightarrow{+2} v_4$.
From this sequence, we get

$$rn((P_7^{+-+})_2) \leq spanf = f(x_{13}) = f(x_1) + \sum_{k=2}^{13}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 2(1) + 10(2) = 23. \quad (29)$$

For $n = 8$.
$x_1 = v_1 \xrightarrow{+1} v_6 \xrightarrow{+2} v_2 \xrightarrow{+1} v_7 \xrightarrow{+2} v_3 \xrightarrow{+1} v_8 \xrightarrow{+2} v_5 \xrightarrow{+2} e_1 \xrightarrow{+2} e_2 \xrightarrow{+2} e_3 \xrightarrow{+2} e_4 \xrightarrow{+2} e_5 \xrightarrow{+2} e_6 \xrightarrow{+2} e_7 \xrightarrow{+2} v_4 = x_{2n-1}$.
From this sequence, we get

$$rn((P_8^{+-+})_2) \leq spanf = f(x_{15}) = f(x_1) + \sum_{k=2}^{15}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 3(1) + 11(2) = 26. \quad (30)$$

For $n = 9$.
$x_1 = v_4 \xrightarrow{+1} v_9 \xrightarrow{+2} v_1 \xrightarrow{+1} v_6 \xrightarrow{+2} v_2 \xrightarrow{+1} v_7 \xrightarrow{+2} v_3 \xrightarrow{+1} v_8 \xrightarrow{+2} v_5 \xrightarrow{+2} e_1 \xrightarrow{+2} e_2 \xrightarrow{+2} e_3 \xrightarrow{+2} e_4 \xrightarrow{+2} e_5 \xrightarrow{+2} e_6 \xrightarrow{+2} e_7 \xrightarrow{+2} e_8 = x_{2n-1}$.
From this sequence, we get

$$rn((P_9^{+-+})_2) \leq spanf = f(x_{17}) = f(x_1) + \sum_{k=2}^{17}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 4(1) + 12(2) = 29. \quad (31)$$

For $n = 10$.

$x_1 = v_4 \xrightarrow{+1} v_9 \xrightarrow{+1} v_1 \xrightarrow{+1} v_6 \xrightarrow{+2} e_1 \xrightarrow{+2} v_5 \xrightarrow{+1} v_{10} \xrightarrow{+2} v_2 \xrightarrow{+1} v_7 \xrightarrow{+2} v_3 \xrightarrow{+1} v_8 \xrightarrow{+2} e_2 \xrightarrow{+2} e_3 \xrightarrow{+2} e_4 \xrightarrow{+2} e_5 \xrightarrow{+2} e_6 \xrightarrow{+2} e_7 \xrightarrow{+2} e_8 \xrightarrow{+2} e_9 = x_{2n-1}$.

From this sequence, we get

$$rn((P_{10}^{+-+})_2) \leq spanf = f(x_{19}) = f(x_1) + \sum_{k=2}^{19}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 6(1) + 12(2) = 31. \quad (32)$$

For $n = 11$. $x_1 = v_4 \xrightarrow{+1} v_9 \xrightarrow{+1} v_1 \xrightarrow{+1} v_6 \xrightarrow{+1} v_{11} \xrightarrow{+1} v_3 \xrightarrow{+1} v_8 \xrightarrow{+2} e_1 \xrightarrow{+2} v_5 \xrightarrow{+1} v_{10} \xrightarrow{+1} v_2 \xrightarrow{+1} v_7 \xrightarrow{+2} e_2 \xrightarrow{+2} e_3 \xrightarrow{+2} e_4 \xrightarrow{+2} e_5 \xrightarrow{+2} e_6 \xrightarrow{+2} e_7 \xrightarrow{+2} e_8 \xrightarrow{+2} e_9 \xrightarrow{+2} e_{10} = x_{2n-1}$.

From this sequence, we get

$$rn((P_{11}^{+-+})_2) \leq spanf = f(x_{21}) = f(x_1) + \sum_{k=2}^{21}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 9(1) + 11(2) = 32. \quad (33)$$

For $n = 12$.

$x_1 = v_5 \xrightarrow{+1} v_{10} \xrightarrow{+1} v_2 \xrightarrow{+1} v_7 \xrightarrow{+1} v_{12} \xrightarrow{+1} v_4 \xrightarrow{+1} v_9 \xrightarrow{+1} v_1 \xrightarrow{+1} v_6 \xrightarrow{+1} v_{11} \xrightarrow{+1} v_3 \xrightarrow{+1} v_8 \xrightarrow{+2} e_1 \xrightarrow{+2} e_2 \xrightarrow{+2} e_3 \xrightarrow{+2} e_4 \xrightarrow{+2} e_5 \xrightarrow{+2} e_6 \xrightarrow{+2} e_7 \xrightarrow{+2} e_8 \xrightarrow{+2} e_9 \xrightarrow{+2} e_{10} \xrightarrow{+2} e_{11} = x_{2n-1}$.

From this sequence, we get

$$rn((P_{12}^{+-+})_2) \leq spanf = f(x_{23}) = f(x_1) + \sum_{k=2}^{23}[f(x_i) - f(x_{i-1})]$$
$$= 1 + 11(1) + 11(2) = 34. \quad (34)$$

When, $n \geq 13$, define $f : V(G) \to Z^+$ as

1. $f(v_1) = \lfloor \frac{n+1}{5} \rfloor + 1$, $f(v_2) = \lfloor 2\left(\frac{n-1}{5}\right) \rfloor + 2$, $f(v_3) = \lfloor 3\left(\frac{n-2}{5}\right) \rfloor + 3$, $f(v_4) = 1$, $f(v_5) = n + 1 - \lfloor \frac{n}{5} \rfloor$
2. $f(v_i) = f(v_{i-4}) + 1$ for $6 \leq i \leq n$
3. $f(e_1) = \max\{f(v_i) : 1 \leq i \leq n\} + 2 = n + 2$
4. $f(e_{i+1}) = f(e_i) + 2$ for $1 \leq i \leq n - 2$

Clearly f is a radio labeling by Lemma (5.1). In fact $|f(v_i) - f(v_j)| \geq 1$ if $j \geq i \pm 5$, (that is $d(v_i, v_j) = 3$) , $|f(v_i) - f(v_j)| \geq 2$ if $j \geq i \pm 3$, (that is $d(v_i, v_j) = 2$), $|f(v_i) - f(v_j)| \geq 3$

if $j \geq i \pm 1$ (that is $d(v_i, v_j) = 1$). $|f(v_i) - f(e_j)| \geq n + 2$ (since $d(v_i, e_j) = 2, \forall i, j$). Now $|f(e_i) - f(e_j)| \geq 2$ if $j = i \pm 1, 2$ (that is $d(e_i, e_j) = 2$). $|f(e_i) - f(e_j)| \geq 4$ if $j \geq i \pm 3$ (that is $d(e_i, e_j) = 1$). Hence for all $n \geq 13$;

$$\begin{aligned} rn((P_n^{+-+})_2) \leq span f = f(e_{n-1}) &= f(e_{n-2}) + 2 \\ &= [f(e_{n-3}) + 2] + 2) \\ &\vdots \\ &= f(e_1) + 2(n-2) \\ &= n + 2 + (n-2)(2) = 3n - 2. \end{aligned} \tag{35}$$

Now, inequalities (27)-(35) and inequality (9) together prove Theorem 3.7.

Case 7: $xyz = ++-$.

For $n = 4$, it follows from Figure 18 that

$$rn((P_4^{++-})_2) \leq 12 \tag{36}$$

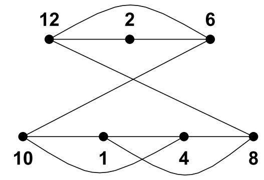

Figure 18: A radio labeling of $(P_4^{++-})_2$

Now, inequality (10), inequality (36) and Lemma 5.2 together prove Theorem 3.8.

Acknowledgement

The authors would like to thank the anonymous referees for their helpful suggestions.

References

[1] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, (1990).

[2] G. Chartrand, D. Erwin, F. Harary and P. Zhang, *Radio labelings of graphs*, Bull. Inst. Combin. Appl., 33 (2001), 77-85.

[3] G. Chartrand, D. Erwin and P. Zhang, *A graph labeling problem suggested by FM Channel Restrictions*, Bulletin of the Inst. Combin. Appl. 43 (2005), 43-57.

[4] P. Devadasa Rao, B. Sooryanarayana and Chandru Hegde, *Radio number of fourth power of a path*, International Journal of Combinatorial graph theory and application, 3 (2010), no. 2, 105-127.

[5] P. Devadas Rao, B. Sooryanarayana and C. Hegde, *Radio Number of k^{th} Power of a Path*, Advanced study in Contemporary Mathematics, 28 (2018), no. 3, 459-478.

[6] W. K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE, 68 (1980), 1497-1514.

[7] F. Harary, *Graph Theory*, Addison-Wesley, (1969).

[8] D. D. F. Liu, *Radio Number for Trees*, Discrete Mathematics, 308 (2008), 1153-1164.

[9] D. D. F. Liu and M. Xie, *Radio number for square of cycles*, Congr, Numer., 169 (2004), 105-125.

[10] D. D. F. Liu and M. Xie, *Radio number of square paths*, Ars Comb., 90 (2009), 307-319.

[11] D. D. F. Liu and X. Zhu, *Multilevel Distance Labelings for paths and cycles*, SIAM J. Discrete Math., 19 (2005), 610-621.

[12] B. Sooryanarayana and Raghunath P., *Radio labeling of cube of a cycle*, Far East J. Appl. Math, 29 (2007), no. 1, 113-147.

[13] B. Sooryanarayana and Raghunath P., *Radio labeling of fourth power of a cycle*, Journal of Applied Mathematical Analysis and Applications, 3 (2007), no. 2, 195-228.

[14] B. Sooryanarayana, Vishu kumar M. and Manjula K., *Radio number of cube of a path*, International J. Math. Combin., 1 (2010), no. 1, 5-29.

[15] B. Wu and J. Meng, *Basic properies of total transformation graphs*, J. Math. Study, 34 (2001), no. 2, 109-116.

[16] S. Yogalakshmi, B. Sooryanarayana, Ramya, *Radio number of transformation graphs of a path*, J. Appl. Math. and Informatics, 35 (2017), no. 1 - 2, 59 - 74.

[17] P. Zhang, *Radio Labelings of Cycles*, Ars Combin., 65 (2002), 21-32.