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Some Inequalities for the Forgotten Topological Index of a Line Graph

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Abstract

Let G be a simple connected graph with n vertices and m edges, and let $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m)$ be a sequence of edge degrees of G. Denote by $EF = \sum_{i=1}^{m} d(e_i)^3$ reformulated forgotten index of graph G. Lower and upper bounds for EF are obtained.

Key words: Topological indices (of graph), line graph, forgotten topological index.

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1 Introduction

Let $G = (V, E)$, $V = \{1, 2, \dots, n\}$, $E = \{e_1, e_2, \dots, e_m\}$, be a simple connected graph with n vertices and m edges. Denote by $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(i)$, and $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m)$, sequences of its vertex and edge degrees, respectively. We will use the following notation: $\Delta_{e_1} = d(e_1) + 2$, $\Delta_{e_2} = d(e_2) + 2$, $\delta_{e_1} = d(e_m) + 2$, $\delta_{e_2} = d(e_{m-1}) + 2$. If i-th and j-th vertices (e_i and e_j edges) are adjacent, we write $i \sim j$ ($e_i \sim e_j$). As usual, $L(G)$ denotes a line graph of graph G.

A topological index for a graph is a numerical quantity which is invariant under automorphisms of the graph. The simplest topological indices are the number of vertices and edges of graph. Two vertex-degree-based topological indices, known as the first and the second Zagreb indices, M_1 and M_2, are defined as [8]:

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{i \sim j} d_i d_j.$$

As shown in [5], the first Zagreb index M_1 can also be expressed as

$$M_1 = \sum_{i \sim j} (d_i + d_j). \tag{1}$$

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A so called forgotten topological index, F, is defined as [7] (see also [9]):

$$F = F(G) = \sum_{i=1}^{n} d_i^3.$$

By analogy to M_1, the invariant F can be written in the following way

$$F = \sum_{i\sim j}(d_i^2 + d_j^2) = \sum_{i\sim j}(d_i + d_j)^2 - 2M_2. \tag{2}$$

Details on the topological indices M_1, M_2 and F can be found in [2, 3, 5, 6, 9, 10, 11, 12, 16, 17].

The first and the second reformulated Zagreb indices, EM_1 and EM_2, are defined as [15]

$$EM_1 = EM_1(G) = \sum_{i=1}^{m} d(e_i)^2 \quad \text{and} \quad EM_2 = EM_2(G) = \sum_{e_i\sim e_j} d(e_i)d(e_j).$$

In essence, EM_1 and EM_2 are not new topological indices, as these correspond to the first and the second Zagreb indices of a line graph $L(G)$ of graph G. Therefore, we have $EM_1(G) = M_1(L(G))$ and $EM_2(G) = M_2(L(G))$. According to this, we can define reformulated forgotten index of a line graph as (see [18])

$$EF = EF(G) = \sum_{i=1}^{m} d(e_i)^3. \tag{3}$$

Todeschini et all [20] (see also [21]) introduced multiplicative variants of the first and the second Zagreb indices, Π_1 and Π_2, as

$$\Pi_1 = \Pi_1(G) = \prod_{i=1}^{n} d_i^2 \quad \text{and} \quad \Pi_2 = \Pi_2(G) = \prod_{i\sim j} d_i d_j.$$

Multiplicative sum first Zagreb index, Π_1^*, was defined in [6] as

$$\Pi_1^* = \Pi_1^*(G) = \prod_{i\sim j}(d_i + d_j). \tag{4}$$

In this paper we are interested in topological index EF. We prove some inequalities involving lower and upper bounds for EF.

2 Preliminaries

In this section we recall some discrete inequalities for real number sequences that will be used in proofs of theorems.

Let $I = \{1, 2, \ldots, m\}$ and $I_2 = \{1, m\}$ be index sets. If $a = (a_i)$, $i = 1, 2, \ldots, m$, is a real number sequence and $p = (p_i)$, $i = 1, 2, \ldots, m$, is positive real number sequence, then a weighted mean of numbers $a_1, a_2, \ldots, a_m$ is defined as

$$M_1(a, p; I) = \frac{\sum_{i=1}^{m} p_i a_i}{\sum_{i=1}^{m} p_i}.$$

Let $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \ldots, m$, be non-negative real number sequences of similar monotonicity and $p = (p_i)$, $i = 1, 2, \ldots, m$, be positive real number sequence. If pairs

$$(M_1(a, p; I - I_2), M_1(a, p; I_2)) \quad \text{and} \quad (M_1(b, p; I - I_2), M_1(b, p; I_2))$$

are similarly ordered, the following inequality is valid [13]

$$\sum_{i=1}^{m} p_i \sum_{i=1}^{m} p_i a_i b_i - \sum_{i=1}^{m} p_i a_i \sum_{i=1}^{m} p_i b_i \geq \frac{p_1 p_m (a_1 - a_m)(b_1 - b_m)}{p_1 + p_m} \sum_{i=1}^{m} p_i, \tag{5}$$

with equality if and only if $a_2 = \cdots = a_{m-1} = (a_1 + a_m)/2$ or $b_2 = \cdots = b_{m-1} = (b_1 + b_m)/2$.

Let $p = (p_i)$, and $a = (a_i)$, $i = 1, 2, \ldots, m$, be two positive real number sequences with the properties

$$\sum_{i=1}^{m} p_i = 1, \quad 0 < r \leq a_i \leq R < +\infty, \quad i = 1, 2, \ldots, m.$$

In [19] the following inequality was proven

$$\sum_{i=1}^{m} p_i a_i + rR \sum_{i=1}^{m} \frac{p_i}{a_i} \leq r + R. \tag{6}$$

Equality in (6) holds if and only if $R = a_1 = \cdots = a_m = r$, or for arbitrary k, $1 \leq k \leq m - 1$, holds $R = a_1 = \cdots = a_k \geq a_{k+1} = \cdots = a_m = r$.

Let $a_1 \geq a_2 \geq \cdots \geq a_m > 0$ be positive real number sequence. In [4] it was proven

$$\sum_{i=1}^{m} a_i - m \left(\prod_{i=1}^{m} a_i \right)^{\frac{1}{m}} \geq \left(\sqrt{a_1} - \sqrt{a_m} \right)^2, \tag{7}$$

with equality if and only if $a_2 = \cdots = a_{m-1} = \sqrt{a_1 a_m}$, and

$$\frac{\sum_{i=1}^{m} a_i}{m\left(\prod_{i=1}^{m} a_i\right)^{\frac{1}{m}}} \geq \left(\frac{a_1 + a_m}{2\sqrt{a_1 a_m}}\right)^{\frac{2}{m}}, \tag{8}$$

with equality if and only if $a_2 = \cdots = a_{m-1} = (a_1 + a_m)/2$.

Let $a = (a_i)$, and $b = (b_i)$, $i = 1, 2, \ldots, m$, be two positive real number sequences with the properties

$$0 < r_1 \leq a_i \leq R_1 < +\infty \quad \text{and} \quad 0 < r_2 \leq b_i \leq R_2 < +\infty.$$

In [1] the following inequality was proven

$$\left| m\sum_{i=1}^{m} a_i b_i - \sum_{i=1}^{m} a_i \sum_{i=1}^{m} b_i \right| \leq m^2 \alpha(m)(R_1 - r_1)(R_2 - r_2), \tag{9}$$

where

$$\alpha(m) = \frac{1}{m}\left\lfloor \frac{m}{2} \right\rfloor \left(1 - \frac{1}{m}\left\lfloor \frac{m}{2} \right\rfloor\right) = \frac{1}{4}\left(1 - \frac{(-1)^{m+1} + 1}{2m^2}\right).$$

For positive real number sequence $a = (a_i)$, $i = 1, 2, \ldots, m$, in [22] (see also [14]) the following inequality was proven

$$m\left(\frac{1}{m}\sum_{i=1}^{m} a_i - \left(\prod_{i=1}^{m} a_i\right)^{\frac{1}{m}}\right) \leq m\sum_{i=1}^{m} a_i - \left(\sum_{i=1}^{m} \sqrt{a_i}\right)^2. \tag{10}$$

3 Main results

In the following theorem we prove the inequality that sets up lower bound for EF in terms of invariants M_1, M_2, F, and parameter m.

Theorem 3.1. Let G be a simple connected graph with $m \geq 2$ edges. Then

$$EF \geq \frac{(F + 2M_2 - 3M_1)^2}{M_1} + 3M_1 - 8m + \frac{\Delta_{e_1}\delta_{e_1}(\Delta_{e_1} - \delta_{e_1})^2}{\Delta_{e_1} + \delta_{e_1}}. \tag{11}$$

Equality holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2} = (\Delta_{e_1} + \delta_{e_1})/2$.

Proof: For $p_i = a_i = b_i = d(e_i) + 2$, $i = 1, 2, \ldots, m$, the inequality (5) transforms into

$$\sum_{i=1}^{m}(d(e_i)+2)\sum_{i=1}^{m}(d(e_i)+2)^3 \geq \left(\sum_{i=1}^{m}(d(e_i)+2)^2\right)^2 + \frac{\Delta_{e_1}\delta_{e_1}(\Delta_{e_1}-\delta_{e_1})^2}{\Delta_{e_1}+\delta_{e_1}}\sum_{i=1}^{m}(d(e_i)+2). \quad (12)$$

Let $e = \{i, j\}$ be an arbitrary edge of graph G. Then $d(e) = d_i + d_j - 2$, and from (1) and (2) we have

$$M_1 = \sum_{i\sim j}(d_i + d_j) = \sum_{i=1}^{m}(d(e_i) + 2),$$

and

$$F + 2M_2 = \sum_{i\sim j}(d_i + d_j)^2 = \sum_{i=1}^{m}(d(e_i) + 2)^2.$$

According to these equalities and (12) it follows

$$\sum_{i=1}^{m}(d(e_i) + 2)^3 \geq \frac{(F + 2M_2)^2}{M_1} + \frac{\Delta_{e_1}\delta_{e_1}(\Delta_{e_1}-\delta_{e_1})^2}{\Delta_{e_1}+\delta_{e_1}}. \quad (13)$$

On the other hand, we have

$$\begin{aligned}\sum_{i=1}^{m}(d(e_i) + 2)^3 &= \sum_{i=1}^{m}\Big(d(e_i)^3 + 6(d(e_i) + 2)^2 - 12(d(e_i) + 2) + 8\Big)\\ &= \sum_{i=1}^{m} d(e_i)^3 + 6(F + 2M_2) - 12M_1 + 8m,\end{aligned}$$

that is

$$EF = \sum_{i=1}^{m}(d(e_i) + 2)^3 - 6(F + 2M_2) + 12M_1 - 8m. \quad (14)$$

Combining relations (14) and (13) we get inequality (11).

Equality in (13) holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2} = (\Delta_{e_1} + \delta_{e_1})/2$, therefore equality in (11) holds under the same conditions. ∎

By a similar procedure as in the case of Theorem 3.1, the following statement can be proved.

Theorem 3.2. Let G be a simple connected graph with m edges. If $m \geq 3$, then

$$EF \geq \Delta_{e_1}^3 + \frac{(F + 2M_2 - \Delta_{e_1}^2)^2}{M_1 - \Delta_{e_1}} - 6(F + 2M_2) + 12M_1 - 8m.$$

Equality holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_m) + 2 = \delta_{e_1}$.

If $m \geq 3$, then

$$EF \geq \delta_{e_1}^3 + \frac{(F + 2M_2 - \delta_{e_1}^2)^2}{M_1 - \delta_{e_1}} - 6(F + 2M_2) + 12M_1 - 8m.$$

Equality holds if and only if $L(G)$ is regular graph.

If $m \geq 4$, then

$$EF \geq \Delta_{e_1}^3 + \delta_{e_1}^3 + \frac{(F + 2M_2 - \Delta_{e_1}^2 - \delta_{e_1}^2)^2}{M_1 - \Delta_{e_1} - \delta_{e_1}} - 6(F + 2M_2) + 12M_1 - 8m.$$

Equality holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$.

In the next theorem we prove inequality that establishes an upper bound for EF in terms of invariants M_1, M_2, F, and parameter m.

Theorem 3.3. Let G be a simple connected graph with $m \geq 1$ edges. Then

$$EF \leq (\Delta_{e_1} + \delta_{e_1} - 6)(F + 2M_2) + (12 - \Delta_{e_1}\delta_{e_1})M_1 - 8m. \tag{15}$$

Equality holds if and only if $L(G)$ is a regular graph or for arbitrary k, $1 \leq k \leq m - 1$, holds $\Delta_{e_1} = d(e_1) + 2 = \cdots = d(e_k) + 2 \geq d(e_{k+1}) + 2 = \cdots = d(e_m) + 2 = \delta_{e_1}$.

Proof: For

$$p_i = \frac{(d(e_i) + 2)^2}{\sum_{i=1}^{m}(d(e_i) + 2)^2}, \quad a_i = d(e_i) + 2,$$

$i = 1, 2, \ldots, m$, $r = \delta_{e_1} = d(e_m) + 2$, and $R = \Delta_{e_1} = d(e_1) + 2$, the inequality (6) becomes

$$\frac{\sum_{i=1}^{m}(d(e_i) + 2)^3}{\sum_{i=1}^{m}(d(e_i) + 2)^2} + \Delta_{e_1}\delta_{e_1}\frac{\sum_{i=1}^{m}(d(e_i) + 2)}{\sum_{i=1}^{m}(d(e_i) + 2)^2} \leq \Delta_{e_1} + \delta_{e_1},$$

i.e.

$$\sum_{i=1}^{m}(d(e_i) + 2)^3 \leq (\Delta_{e_1} + \delta_{e_1})(F + 2M_2) - \Delta_{e_1}\delta_{e_1}M_1. \tag{16}$$

From (14) and (16) we obtain inequality (15).

Equality in (6) holds if and only if $R = a_1 = \cdots = a_m = r$, or for arbitrary k, $1 \leq k \leq m-1$, holds $R = a_1 = \cdots = a_k \geq a_{k+1} = \cdots = a_m = r$. Therefore, equality in (15) holds if and only if $L(G)$ is regular graph or for arbitrary k, $1 \leq k \leq m-1$, holds $\Delta_{e_1} = d(e_1) + 2 = \cdots = d(e_k) + 2 \geq d(e_{k+1}) + 2 = \cdots = d(e_m) + 2 = \delta_{e_1}$. ∎

Corollary 3.4. Let G be a simple connected graph with $m \geq 2$ edges. Then

$$EF \leq \frac{(F+2M_2)^2}{4M_1}\left(\sqrt{\frac{\Delta_{e_1}}{\delta_{e_1}}} + \sqrt{\frac{\delta_{e_1}}{\Delta_{e_1}}}\right)^2 - 6(F+2M_2) + 12M_1 - 8m. \tag{17}$$

Equality holds if $L(G)$ is regular.

Proof: The above inequality is obtained from (14) and

$$2\sqrt{\Delta_{e_1}\delta_{e_1}M_1\sum_{i=1}^{m}(d(e_i)+2)^3} \leq \sum_{i=1}^{m}(d(e_i)+2)^3 + \Delta_{e_1}\delta_{e_1}M_1 \leq (\Delta_{e_1}+\delta_{e_1})(F+2M_2).$$

∎

In the same way as in Theorem 3.3 we can prove the following result.

Theorem 3.5. Let G be a simple connected graph with m edges. If $m \geq 3$, then

$$EF \leq \Delta_{e_1}^3 + (\Delta_{e_2}+\delta_{e_1})(F+2M_2-\Delta_{e_1}^2) - \Delta_{e_2}\delta_{e_1}(M_1-\Delta_{e_1}) - 6(F+2M_2) + 12M_1 - 8m.$$

Equality holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_m) + 2 = \delta_{e_1}$ or for arbitrary k, $2 \leq k \leq m-1$, holds $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_k) + 2 \geq d(e_{k+1}) + 2 = \cdots = d(e_m) + 2 = \delta_{e_1}$.

If $m \geq 3$, then

$$EF \leq \delta_{e_1}^3 + (\Delta_{e_1}+\delta_{e_2})(F+2M_2-\delta_{e_1}^2) - \Delta_{e_1}\delta_{e_2}(M_1-\delta_{e_1}) - 6(F+2M_2) + 12M_1 - 8m.$$

Equality holds if and only if $L(G)$ is regular or for arbitrary k, $1 \leq k \leq m-2$, holds $\Delta_{e_1} = d(e_1) + 2 = \cdots = d(e_k) + 2 \geq d(e_{k+1}) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$.

If $m \geq 4$, then

$$\begin{aligned} EF \leq\ & \Delta_{e_1}^3 + \delta_{e_1}^3 + (\Delta_{e_2}+\delta_{e_2})(F+2M_2-\Delta_{e_1}^2-\delta_{e_1}^2) \\ & - \Delta_{e_2}\delta_{e_2}(M_1-\Delta_{e_1}-\delta_{e_1}) - 6(F+2M_2) + 12M_1 - 8m. \end{aligned}$$

Equality holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$ or for arbitrary

k, $2 \leq k \leq m-2$, holds $\Delta_{e_2} = d(e_2)+2 = \cdots = d(e_k)+2 \geq d(e_{k+1})+2 = \cdots = d(e_{m-1})+2 = \delta_{e_2}$.

In the following theorem we determine lower bound for EF in terms of parameters m, Δ_{e_1}, δ_{e_1}, and graph invariants M_1, M_2, F, and Π_1^*.

Theorem 3.6. Let G be a simple connected graph with $m \geq 2$ edges. Then

$$EF \geq m\left(\Pi_1^*\right)^{\frac{3}{m}} - 6(F+2M_2) + 12M_1 - 8m + \left((\Delta_{e_1})^{\frac{3}{2}} - (\delta_{e_1})^{\frac{3}{2}}\right)^2. \tag{18}$$

Equality holds if and only if $\Delta_{e_2} = d(e_2)+2 = \cdots = d(e_{m-1})+2 = \delta_{e_2} = \sqrt{\Delta_{e_1}\delta_{e_1}}$.

Proof: For $a_i = (d(e_i)+2)^3$, $i = 1, 2, \ldots, m$, the inequality (7) becomes

$$\sum_{i=1}^{m}(d(e_i)+2)^3 \geq m\left(\prod_{i=1}^{m}(d(e_i)+2)^3\right)^{\frac{1}{m}} + \left((\Delta_{e_1})^{\frac{3}{2}} - (\delta_{e_1})^{\frac{3}{2}}\right)^2.$$

Since

$$\Pi_1^* = \prod_{i\sim j}(d_i+d_j) = \prod_{i=1}^{m}(d(e_i)+2),$$

this inequality transforms into

$$\sum_{i=1}^{m}(d(e_i)+2)^3 \geq m\left(\Pi_1^*\right)^{\frac{3}{m}} + \left((\Delta_{e_1})^{\frac{3}{2}} - (\delta_{e_1})^{\frac{3}{2}}\right)^2. \tag{19}$$

Now, from (14) and (19) we obtain (18).

Equality in (7) holds if and only if $a_2 = \cdots = a_{m-1} = \sqrt{a_1 a_m}$, therefore equality in (18) is attained if and only if $\Delta_{e_2} = d(e_2)+2 = \cdots = d(e_{m-1})+2 = \delta_{e_2} = \sqrt{\Delta_{e_1}\delta_{e_1}}$. ∎

By a similar procedure as in the case of Theorem 3.6, the following statement can be proved.

Theorem 3.7. Let G be a simple connected graph with m edges. If $m \geq 3$, then

$$EF \geq \Delta_{e_1}^3 + (m-1)\left(\frac{\Pi_1^*}{\Delta_{e_1}}\right)^{\frac{3}{m-1}} - 6(F+2M_2) + 12M_1 - 8m + \left((\Delta_{e_2})^{\frac{3}{2}} - (\delta_{e_1})^{\frac{3}{2}}\right)^2.$$

Equality holds if and only if $d(e_3)+2 = \cdots = d(e_{m-1})+2 = \delta_{e_2} = \sqrt{\Delta_{e_2}\delta_{e_1}}$.

If $m \geq 3$, then

$$EF \geq \delta_{e_1}^3 + (m-1)\left(\frac{\Pi_1^*}{\delta_{e_1}}\right)^{\frac{3}{m-1}} - 6(F+2M_2) + 12M_1 - 8m + \left((\Delta_{e_1})^{\frac{3}{2}} - (\delta_{e_2})^{\frac{3}{2}}\right)^2.$$

Equality holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-2}) + 2 = \sqrt{\Delta_{e_1}\delta_{e_2}}$.

If $m \geq 4$, then

$$EF \geq \Delta_{e_1}^3 + \delta_{e_1}^3 + (m-2)\left(\frac{\Pi_1^*}{\Delta_{e_1}\delta_{e_1}}\right)^{\frac{3}{m-2}} - 6(F+2M_2) + 12M_1 - 8m + \left((\Delta_{e_2})^{\frac{3}{2}} - (\delta_{e_2})^{\frac{3}{2}}\right)^2.$$

Equality holds if and only if $d(e_3) + 2 = \cdots = d(e_{m-2}) + 2 = \sqrt{\Delta_{e_2}\delta_{e_2}}$.

Similarly as for Theorem 3.6, according to inequality (8) we obtain the following statement.

Theorem 3.8. Let G be a simple connected graph with $m \geq 2$ edges. Then

$$EF \geq m\,(\Pi_1^*)^{\frac{3}{m}}\left(\frac{\Delta_{e_1}^3 + \delta_{e_1}^3}{2\sqrt{\Delta_{e_1}^3\delta_{e_1}^3}}\right)^{\frac{2}{m}} - 6(F + 2M_2) + 12M_1 - 8m.$$

Equality holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2} = ((\Delta_{e_1}^3 + \delta_{e_1}^3)/2)^{1/3}$.

In the following theorem we determine an upper bound for EF in terms of parameters m, Δ_{e_1}, δ_{e_1}, and graph invariants M_1, M_2, F, and Π_1^*.

Theorem 3.9. Let G be a simple connected graph with $m \geq 2$ edges. Then

$$EF \leq m\,(\Pi_1^*)^{\frac{3}{m}} - 6(F + 2M_2) + 12M_1 - 8m + m^2\alpha(m)\left((\Delta_{e_1})^{\frac{3}{2}} - (\delta_{e_1})^{\frac{3}{2}}\right)^2. \qquad (20)$$

Equality holds if and only if $L(G)$ is a regular graph.

Proof: For $a_i = b_i = (d(e_i) + 2)^{3/2}$, $i = 1, 2, \ldots, m$, $R_1 = R_2 = \Delta_{e_1}^{\frac{3}{2}}$, $r_1 = r_2 = \delta_{e_1}^{\frac{3}{2}}$, the inequality (9) becomes

$$m\sum_{i=1}^{m}(d(e_i) + 2)^3 - \left(\sum_{i=1}^{m}(d(e_i) + 2)^{\frac{3}{2}}\right)^2 \leq m^2\alpha(m)\left((\Delta_{e_1})^{\frac{3}{2}} - (\delta_{e_1})^{\frac{3}{2}}\right)^2. \qquad (21)$$

For $a_i = b_i = (d(e_i) + 2)^3$, $i = 1, 2, \ldots, m$, the inequality (10) becomes

$$\left(\sum_{i=1}^{m}(d(e_i) + 2)^{\frac{3}{2}}\right)^2 \leq (m-1)\sum_{i=1}^{m}(d(e_i) + 2)^3 + m\,(\Pi_1^*)^{\frac{3}{m}}. \qquad (22)$$

According to (21) and (22) we get

$$\sum_{i=1}^{m}(d(e_i)+2)^3 \le m\,(\Pi_1^*)^{\frac{3}{m}} + m^2\alpha(m)\left((\Delta_{e_1})^{\frac{3}{2}} - (\delta_{e_1})^{\frac{3}{2}}\right)^2. \tag{23}$$

Now, from (23) and (14) we get (20).

Equality in (10) holds if and only if $a_1 = \cdots = a_m$, therefore equality in (22), i.e. in (20), is attained if and only if $\Delta_{e_1} = d(e_1)+2 = \cdots = d(e_m)+2 = \delta_{e_1}$. ∎

Let $\Delta = d_1$ and $\delta = d_n$. Then we have $2\delta \le \delta_{e_1} \le \Delta_{e_1} \le 2\Delta$. Also, for any m holds $\alpha(m) \le 1/4$. Therefore we have the following corollary of Theorem 3.9.

Corollary 3.10. Let G be a simple connected graph with $n \ge 2$ vertices and m edges. Then

$$EF \le m\,(\Pi_1^*)^{\frac{3}{m}} - 6(F+2M_2) + 12M_1 - 8m + 2m^2\left(\Delta^{\frac{3}{2}} - \delta^{\frac{3}{2}}\right)^2.$$

Equality holds if and only if G is a regular graph.

Since $F \ge 2M_2$ we get the next corollary.

Corollary 3.11. Let G be a simple connected graph with $n \ge 2$ vertices and m edges. Then

$$EF \le m\,(\Pi_1^*)^{\frac{3}{m}} - 24M_2 + 12M_1 - 8m + 2m^2\left(\Delta^{\frac{3}{2}} - \delta^{\frac{3}{2}}\right)^2.$$

Equality holds if and only if G is a regular graph.

Similarly as in case of Theorem 3.9, the following result can be proved.

Theorem 3.12. Let G be a simple connected graph with m edges. If $m \ge 3$, then

$$EF \le \Delta_{e_1}^3 + (m-1)\left(\frac{\Pi_1^*}{\Delta_{e_1}}\right)^{\frac{3}{m-1}} - 6(F+2M_2) + 12M_1 - 8m + (m-1)^2\alpha(m-1)\left((\Delta_{e_2})^{\frac{3}{2}} - (\delta_{e_1})^{\frac{3}{2}}\right)^2.$$

Equality holds if and only if $\Delta_{e_2} = d(e_2)+2 = \cdots = d(e_m)+2 = \delta_{e_1}$.

If $m \ge 3$, then

$$EF \le \delta_{e_1}^3 + (m-1)\left(\frac{\Pi_1^*}{\delta_{e_1}}\right)^{\frac{3}{m-1}} - 6(F+2M_2) + 12M_1 - 8m + (m-1)^2\alpha(m-1)\left((\Delta_{e_1})^{\frac{3}{2}} - (\delta_{e_2})^{\frac{3}{2}}\right)^2.$$

Equality holds if and only if $L(G)$ is a regular graph.

If $m \geq 4$, then

$$EF \leq \Delta_{e_1}^3 + \delta_{e_1}^3 + (m-2)\left(\frac{\Pi_1^*}{\Delta_{e_1}\delta_{e_1}}\right)^{\frac{3}{m-2}} - 6(F + 2M_2) + 12M_1 - 8m$$
$$+ (m-2)^2\alpha(m-2)\left((\Delta_{e_2})^{\frac{3}{2}} - (\delta_{e_2})^{\frac{3}{2}}\right)^2.$$

Equality holds if and only if $\Delta_{e_2} = d(e_2) + 2 = \cdots = d(e_{m-1}) + 2 = \delta_{e_2}$.

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