International Journal of Applied Graph Theory
Vol.5, No.1 (2021), 10 - 24.
$ISSN(Online) : 2456 - 7884$

New Class of k-equitable Symmetric Trees

N. Shanmugapriya
Mathematics Section, Department of Information Technology
University of Technology and Applied Sciences-Ibri
Sultanate of Oman
shanmugapriya.vec@gmail.com

Abstract

In 1990 Cahit [2] introduced k-equitable labeling as a generalization of graceful labeling. For a graph $G(V, E)$ and for a positive integer $k \geq 2$, a function f defined from the vertex set of G to $\{0, 1, 2, \ldots, k - 1\}$ is called k-equitable if every edge uv is assigned the label $|f(u) - f(v)|$, then the number of vertices labeled i and the number of vertices labeled j differ by at most 1 and the number of edges labeled i and the number of edges labeled j differ by at most 1, for i, j, $0 \leq i < j \leq k - 1$. Note that a graph G with m edges is a graceful if and only if it is an $(m + 1)$-equitable. In 1990, Cahit [2] also conjectured that every tree is k-equitable for any $k \geq 2$. This conjecture is equivalent to the celebrated graceful tree conjecture when k is the number of vertices of the tree. Motivated by the Cahit's k-Equitable Tree Conjecture and its relevance to the Graceful Tree Conjecture, here in this paper we show a new family of tree called first two levels degree specific symmetric tree admits k-equitable labeling for $k \geq 2$.

Key words: k-equitable labeling, k-equitable trees, first two levels degree specific symmetric trees.

2010 Mathematics Subject Classification : 05C78

1 Introduction

In 1963, Ringel [7] conjectured that K_{2m+1}, the complete graph on $2m + 1$ vertices, can be decomposed into $2m + 1$ isomorphic copies of a given tree with m-edges. In 1965, Kotzig [6] conjectured that the complete graph K_{2m+1}, can be cyclically decomposed into $2m + 1$ copies of a given tree with m edges. To settle the above two conjectures, in 1967, Rosa [8] introduced a hierarchical series of labeling called ρ, σ, β and α-valuations. Later, Golomb [4] called β-valuation as graceful and now this is the term most widely used.

A graceful labeling of a graph G with m edges and vertex set V is an injection $f : V(G) \to \{0, 1, 2, \ldots, m\}$ with the property that the resulting edge labels are also distinct where an edge

$*$ Corresponding Author: N. Shanmugapriya
Ψ Received on October 22, 2020 / Revised on January 09, 2021 / Accepted on January 22, 2021

incident with vertices u and v is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. In his classical paper [8], Rosa proved the theorem that, if a tree T with m edges has a decomposition into $2m + 1$ copies of T.

From Rosa's theorem it follows that both Ringel and Kotzig's conjectures are true if every tree is graceful. This led to the birth of the popular Ringel-Kotzig-Rosa conjecture popularly called the Graceful Tree Conjecture: "All trees are graceful".

As the Graceful Tree Conjecture is a hard problem to settle, and the characterization of graceful graphs are extremely hard to understand, different generalization on graceful labeling were introduced and studied.

One such generalization of graceful labeling is a k-equitable labeling, which was introduced by Cahit [2] in the year 1990. In the k-equitable labeling, the vertex and the edge labels are distributed as evenly as possible and it is defined more precisely in the following way.

For a graph $G(V, E)$ and for a positive integer $k \geq 2$, a function $f : V(G) \to \{0, 1, 2, \ldots, k-1\}$ is called a k-equitable labeling, if f and its induced edge labeling function $f^* : E(G) \to \{0, 1, 2, \ldots, k-1\}$ defined by $f^*(e = uv) = |f(u) - f(v)|$ satisfying the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_{f^*}(i) - e_{f^*}(j)| \leq 1$ respectively, for i, j, $0 \leq i < j \leq k-1$ where $v_f(i)$ and $e_f(i)$ denote the number of vertices and the number of edges having the label i under f and f^* respectively.

Cahit [1] proved that every tree is 2-equitable. The 2-equitable labeling is called popularly as cordial labeling. Speyer and Szaniszlo [10] proved that every tree is 3-equitable. Szaniszlo [9] proved that every path, every star are k-equitable and $K_{2,n}$ is k-equitable if and only if $n \equiv k - 1 \pmod{k}$, $n \equiv 0, 1, 2, \ldots, \lfloor \frac{k}{2} \rfloor - 1 \pmod{k}$, or $n = \lfloor \frac{k}{2} \rfloor$ and k is odd. For an exhaustive survey on k-equitable trees refer the excellent dynamic survey by Gallian [5].

In 1990, Cahit [3] conjectured that every tree is k-equitable for any $k \geq 2$. This conjecture is equivalent to the celebrated Graceful Tree Conjecture when k is the number of vertices of the tree. One possible approach to prove the popular Graceful Tree Conjecture is to prove the more general k-Equitable Tree Conjecture. Inspired by this general approach, in this paper, we show that a new class of trees called first two levels degree specific symmetric trees are k-equitable for $k \geq 2$, where the first two levels degree specific symmetric tree is defined below.

The first two levels degree specific symmetric tree is defined to be the rooted tree with degree of the root is $d_0 = m_0 k$, where $k \geq 2$, $m_0 \geq 1$. The degree of each of the vertices in the first level is any odd number $d_1 = 2r_1 + 1$, $r_1 \geq 0$, and for every level l, $2 \leq l \leq n$, each of the vertices in the l^{th} level has the common degree $d_l \geq 2$.

2 Main Result

In this section we prove our main result in theorem 2.2. To prove our main result we made the following observation.

Observation 2.1. Let T be the first two levels degree specific symmetric tree, then from its definition, the following facts can be ascertained.

1. The number of vertices in level 0 is $|V_0| = 1$.

2. The number of vertices in level 1 is $|V_1| = m_0k$.

3. For ℓ, $2 \leq \ell \leq n$, the number of vertices in the level ℓ is $|V_\ell| = \left[\prod_{i=1}^{\ell-1}(d_i - 1)\right] m_0k$, where $d_0 = m_0k$, $k \geq 2$, $m_0 \geq 1$; $d_1 = 2r_1 + 1$, $r_1 \geq 0$, d_i is the common degree of all the vertices of the level i, for $i, 2 \leq i \leq \ell - 1$.

4. The number of vertices in the tree T is,

$$\begin{aligned}|V(T)| &= |V_0| + |V_1| + \cdots + |V_n|\\ &= 1 + m_0k + m_0k(d_1-1) + m_0k(d_1-1)(d_2-1) + \cdots\\ &\qquad + m_0k(d_1-1)(d_2-1)\dots(d_n-1)\\ &= 1 + m_0k + \left[\sum_{\ell=2}^{n}\left(\prod_{i=1}^{\ell-1}(d_i-1)\right)m_0k\right].\end{aligned}$$

5. The number of edges incident to the root of the tree T is, $E_0 = m_0k$.

6. For $\ell \geq 1$, the number of edges incident to each vertex on level ℓ is E_ℓ, then $E_\ell = \left[\prod_{i=1}^{\ell}(d_i - 1)\right] m_0k$.

7. The number of edges in the tree T is

$$\begin{aligned}|E(T)| &= |E_0| + |E_1| + |E_2| + \cdots + |E_{n-1}|\\ &= m_0k + m_0k(d_1-1) + m_0k(d_1-1)(d_2-1) + \cdots\\ &\qquad + m_0k(d_1-1)(d_2-1)\dots(d_{n-1}-1)\\ &= m_0k + \sum_{\ell=1}^{n-1}\left[\left(\prod_{i=1}^{\ell}(d_i-1)\right)m_0k\right].\end{aligned}$$

Theorem 2.2. If T is a first two levels degree specific symmetric tree such that the root of T has the degree m_0k, where $m_0 \geq 1$ and $k \geq 2$, then T is k-equitable.

Proof: Let T be a first two levels degree specific symmetric tree. Then by the definition of T, the degree of the root is $d_0 = m_0k$, where $k \geq 2$, $m_0 \geq 1$. The degree of each of the vertices in the first level is an odd number $d_1 = 2r_1 + 1$, $r_1 \geq 0$. Then, for every level ℓ, $2 \leq \ell \leq n$, each of the vertices in the ℓ^{th} level has the common degree $d_\ell \geq 2$. From Observation 2.1 we have

$$|V(T)| = 1 + m_0k + \sum_{\ell=2}^{n} \left[\prod_{i=1}^{\ell-1}(d_i - 1)\right] m_0k \text{ and}$$
$$\begin{aligned} |E(T)| &= |E_0| + |E_1| + \cdots + |E_{n-1}| \\ &= m_0k + \sum_{\ell=1}^{n-1}\left[\left(\prod_{i=1}^{\ell}(d_i - 1)\right) m_0k\right] \end{aligned}$$

To prove T is k-equitable for $k \geq 2$, we assign the labels $0, 1, 2, \ldots, k-1$ to the vertices of T in the levelwise.

Step 1. Assigning labels to the vertices in level 0

Assign the label 0 to the root of T.

Step 2. Assigning labels to the vertices in level 1

For convenience, we consider the vertices lying on level 1 as m_0 sets $A_1, A_2, \ldots, A_{m_0}$, each consisting of k vertices, where A_1 consists of first k vertices of level 1 beginning from the leftmost vertex of level 1, A_2 consists of the next k vertices starting from the $(k+1)^{th}$ vertex from the left, A_3 consists of next set of k vertices starting from $(2k+1)^{th}$ vertex from the left and in general A_j th set consists of k vertices starting from $((j-1)k+1)^{th}$ vertex from left to right, for j, $4 \leq j \leq m_0$. Further, for j, $1 \leq j \leq m_0$, we name the k vertices of A_j as $u_{(j-1)k+1}, u_{(j-1)k+2}, \ldots, u_{(j-1)k+k}$. Then for each j, $1 \leq j \leq m_0$, we define the label for the vertices in the set $A_j = \{u_{(j-1)k+1}, u_{(j-1)k+2}, \ldots, u_{(j-1)k+k}\}$ as $f(u_{(j-1)k+i}) = i - 1$, for i, $1 \leq i \leq k$.

From the above labeling, we observe that each of the k-labels, $0, 1, 2, \ldots, k-1$ is assigned m_0 times to the vertices of level 1 of T. As all the vertices in the level 1 of T are adjacent to the root which is labeled as 0. Thus the edges incident to the root get the labels $0, 1, 2, \ldots, k-1$. Further, each of the edge labels $0, 1, 2, \ldots, k-1$ appears exactly at m_0 edges incident to the root.

For convenience of understanding the k-equitable labeling, we define a subset of vertices in each level as well as we consider the edges incident to such vertices. For $1 \leq \ell \leq n$ and for

$0 \leq i \leq k-1$, let $V_\ell(i)$ denote the set of all vertices in the level ℓ of T having the label i. That is, for $1 \leq \ell \leq n$ and for $0 \leq i \leq k-1$, $V_\ell(i) = \{v \in V_\ell \mid f(v) = i\}$. For $0 \leq \ell \leq n$, and for $0 \leq i \leq k-1$, let $E_\ell(i)$ denote the set of all edges incident to each of the vertices in the level ℓ of T having the label i and having the other end vertex in the level $\ell+1$ of T.

That is, for $0 \leq \ell \leq n$ and $0 \leq i \leq k-1$, $E_\ell(i) = \{e \in E_\ell \mid f(e) = i\}$.

It follows from the labeling given in Step 2 that for every pair (i, j), $0 \leq i < j \leq k-1$,

$$|V_1(i)| = |V_1(j)| = m_0 \text{ and } |E_0(i)| = |E_0(j)| = m_0.$$

Step 3. Assigning labels to the vertices in level 2.

Starting from the leftmost vertex u_1 of level 1 and going up to the rightmost vertex u_{m_0k} of level 1, sequentially label the children in the level 2 of each vertex u_p's, for p, $1 \leq p \leq m_0k$ in the level 1 using the following Steps 3.1, 3.2 and 3.3.

Step 3.1.

For a vertex u_p in the level 1, for $1 \leq p \leq m_0k$, consider its $2r_1$ children. Arrange the children of u_p as shown in the following Figure 1.

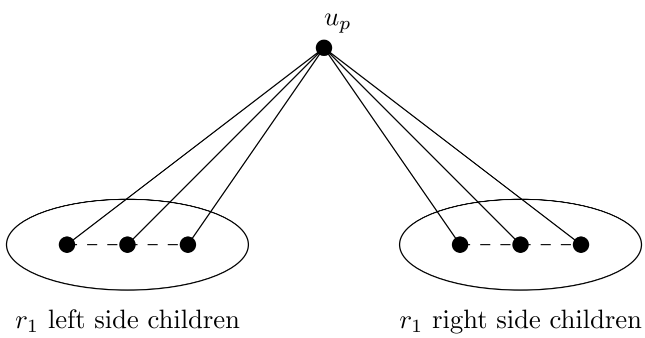

Figure 1: Arrangement of $2r_1$ children of each vertex u_p in level 1 of T

Step 3.2.

If $f(u_p) = i$, where for i, $0 \leq i \leq k-2$, then for each child in the r_1 of the left-side children of u_p assign the label $k-(i+1)$ and for each child in the right side children of u_p assign the label $k-(i+2)$ as shown in the Figure 2.

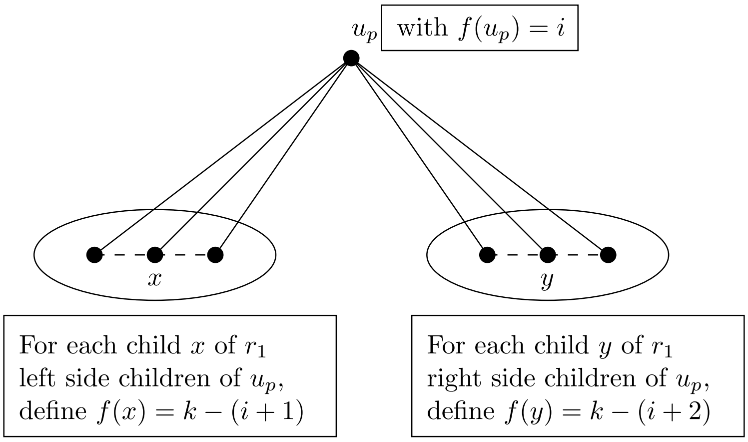

Figure 2: Labeling of the $2r_1$ children of a vertex u_p in the level 1, when $f(u_p) = i$, where for i, $0 \leq i \leq k - 2$

Step 3.3.

If $f(u_p) = k - 1$, then for each child in the r_1 of the left side children of u_p assign the label 0 and for each child in the r_1 of the right side children of u_p assign the label $k - 1$ as shown in the Figure 3.

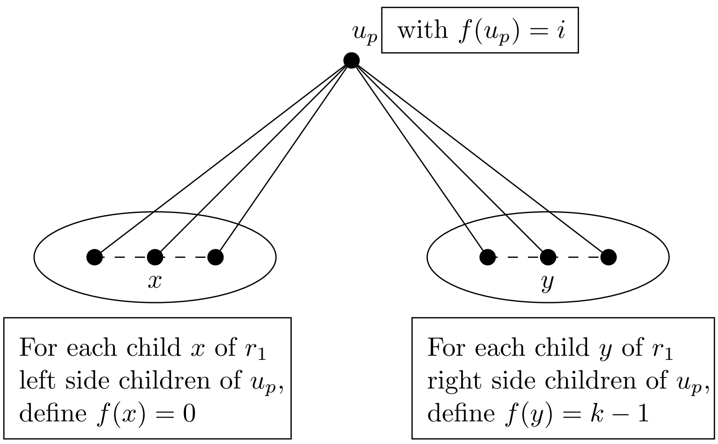

Figure 3: Labeling of the $2r_1$ children of a vertex u_p in the level 1, when $f(u_p) = i$, where for i, $i = k - 1$

The following Table 1 gives the distribution of labels of the vertices in the level 2 of T that are defined in Step 3, (Step 3.1, Step 3.2 and Step 3.3).

Since each label i, $0 \leq i \leq k-1$, appears exactly m_0 times as the vertex label of the vertices of level 1 of T, and as the Table 1 gives the label of the $2r_1$ children of each vertex labeled i,

Table 1: Vertex label of the vertices in the level 2

A vertex having the label i in the level 1	Label of the each child in the left side r_1 children of the vertex in the level 1 labeled i	Label of the each child in the right side r_1 children of the vertex in the level 1 labeled i.
$i = 0$	$k-1$	$k-2$
$i = 1$	$k-2$	$k-3$
$i = 2$	$k-3$	$k-4$
$i = 3$	$k-4$	$k-5$
$\vdots$	$\vdots$	$\vdots$
$i = k-4$	3	2
$i = k-3$	2	1
$i = k-2$	1	0
$i = k-1$	0	$k-1$

in the level 1 of T, we have, for every pair (i, j), $0 \le i < j \le k-1$,

$$|V_2(i)| = |V_2(j)| = 2r_1m_0.$$

The labels of the edges incident to the vertices lying on level 1 and having the other end in level 2 are given in Tables 2, 3, 4 and 5

Observation 1.

From the Table 2 and 3, we observe that if we consider the k-vertices respectively having the labels $0, 1, 2, \ldots, k-1$ and we collect the edges incident to each of the above collected k-vertices, then these edges get the edge labels $0, 1, 2, \ldots, k-1$ and each edge label i, for i, $0 \le i \le k-1$ is obtained exactly $2r_1$ times. Thus, there are m_0 sets of k-vertices having the labels $0, 1, 2, \ldots, k-1$ in the level 1.

Therefore, from Observation 1, for every pair (i, j), $0 \le i < j \le k-1$, we have $|E_1(i)| = |E_1(j)| = 2r_1m_0$.

Table 2: Edge labels of the edges incident to the vertices having the labels $0, 1, 2, \ldots, k-2$ of the level 1 and having the other end in the level 2 corresponding to the case where k is even

A vertex having the label i in the level 1	Label of the r_1 edges incident to the vertex having the label i in the level 1 having the other end vertex in the level 2 having the label $k-(i+1)$	Label of the r_1 edges incident to the vertex having the label i in the level 1 having the other end vertex in the level 2 having the label $k-(i+2)$
$i=0$	$k-1$	$k-2$
$i=1$	$k-3$	$k-4$
$i=2$	$k-5$	$k-6$
$\vdots$	$\vdots$	$\vdots$
$i=\frac{k-4}{2}$	3	2
$i=\frac{k-2}{2}$	1	0
$i=\frac{k}{2}$	1	2
$\vdots$	$\vdots$	$\vdots$
$i=k-4$	$k-7$	$k-6$
$i=k-3$	$k-5$	$k-4$
$i=k-2$	$k-3$	$k-2$

Table 3: Edge labels of the edges incident to the vertices having the labels $k-1$ of the level 1 and having the other end in the level 2 corresponding to the case where k is even

A vertex having the label i in the level 1	Label of the r_1 edges incident to the vertex having the label $k-1$ in the level 1 and having the other end vertex in the level 2 having the label 0	Label of the r_1 edges incident to the vertex having the label $k-1$ in the level 1 having the other end vertex in the level 2 having the label $k-1$
$i=k-1$	$k-1$	0

Table 4: Edge labels of the edges incident to the vertices having the labels $0, 1, 2, \ldots, k-2$ of the level 1 and having the other end in the level 2 corresponding to the case where k is odd

A vertex having the label i in the level 1	Label of the r_1 edges incident to the vertex having the label i in the level 1 and having the other end vertex in the level 2 having the label $k-(i+1)$	Label of the r_1 edges incident to the vertex having the label i in the level 1 and having the other end vertex in the level 2 having the label $k-(i+2)$
$i=0$	$k-1$	$k-2$
$i=1$	$k-3$	$k-4$
$i=2$	$k-5$	$k-6$
$\vdots$	$\vdots$	$\vdots$
$i=\frac{k-3}{2}$	2	1
$i=\frac{k-1}{2}$	0	1
$i=\frac{k+1}{2}$	2	3
$\vdots$	$\vdots$	$\vdots$
$i=k-4$	$k-7$	$k-6$
$i=k-3$	$k-5$	$k-4$
$i=k-2$	$k-3$	$k-2$

Table 5: Edge labels of the edges incident to the vertices having the labels $k-1$ of the level 1 and having the other end in the level 2 corresponding to the case where k is odd

A vertex having the label i in the level 1	Label of the r_1 edges incident to the vertex having the label i in the level 1 having the other end vertex in the level 2 having the label $k-(i+1)$	Label of the r_1 edges incident to the vertex having the label i in the level 1 having the other end vertex in the level 2 having the label $k-(i+2)$
$k-1$	$k-1$	0

We consider the k-vertices respectively having the labels $0, 1, 2, \ldots, k-1$ which are listed in the Table 4 and 5, then, if we collect the edges incident at each of these k-vertices then their incident edges get the edge labels $0, 1, 2, \ldots, k-1$ such that each edge label i, for i, $0 \leq i \leq k-1$ is obtained exactly $2r_1$ times. Since m_0 sets of k vertices having the labels

$0, 1, 2, \ldots, k-1$ in the level 1. Thus, from Observation 1, for every pair (i, j), $0 \leq i < j \leq k-1$, we have $|E_1(i)| = |E_1(j)| = 2r_1m_0$.

Step 4. Assigning labels to the vertices in level 3

To assign labels to the vertices of level 3, we define the following. For $2 \leq \ell \leq n$ and for $0 \leq i \leq k-1$, $ch(V_\ell(i))$ denote the set of all children of every vertex of $V_\ell(i)$. That is, $ch(V_\ell(i)) = \{u \in V_{\ell+1} \mid u \text{ is a child of any } v \in V_\ell(i)\}$. From Step 3, we have $|V_2(i)| = 2r_1m_0$, for i, $0 \leq i \leq k-1$ and $|ch(V_2(i))| = 2r_1m_0(d_2-1)$. Using the following labeling scheme, we assign labels to the $2r_1m_0(d_2-1)$ children of each vertex in $V_2(i)$, for i, $0 \leq i \leq k-1$.

Labeling Scheme 1.

For every vertex $v \in V_2(i)$, for i, $0 \leq i \leq k-2$. First, choose $r_1m_0(d_2-1)$ member of $ch(V_2(i))$ and assign the label $k-(i+1)$ to each of these $r_1m_0(d_2-1)$ member and then choose the remaining $r_1m_0(d_2-1)$ member of $ch(V_2(i))$ and assign the label $k-(i+2)$ to each of these $r_1m_0(d_2-1)$ member.

When $i = k-1$ First choose $r_1m_0(d_2-1)$ member of $ch(V_2(k-1))$ and assign the label 0 to each of these $r_1m_0(d_2-1)$ member and then choose the remaining $r_1m_0(d_2-1)$ member of $ch(V_2(k-1))$ and assign the label $k-1$ to each of these $r_1m_0(d_2-1)$ members. The following table gives the distribution of labels to the vertices of level 3 of T.

Table 6: Vertex label of the vertices in the level 3

Label i of a vertex in the level 2 of T	Label of the each child of the first chosen $r_1m_0(d_2-1)$ children in the $ch(V_2(i))$	Label of the each child for the remaining chosen $r_1m_0(d_2-1)$ children in the $ch(V_2(i))$
$i=0$	$k-1$	$k-2$
$i=1$	$k-2$	$k-3$
$i=2$	$k-3$	$k-4$
$i=3$	$k-4$	$k-5$
$\vdots$	$\vdots$	$\vdots$
$i=k-4$	3	2
$i=k-3$	2	1
$i=k-2$	1	0
$i=k-1$	0	$k-1$

From Table 6, we observe that for every pair (i, j), $0 \leq i < j \leq k-1$, we have $|V_3(i)| = |V_3(j)| = 2r_1m_0(d_2-1)$.

The labels of the edges incident to the vertices lying on level 2 and having the other end in level 3 are given in Tables 7, 8, 9 and 10.

Table 7: Edge labels of the edges incident to the vertices having the label i of the level 2 and having the other end in the level 3 corresponding to the case where k is even

Label i of a vertex in the level 2	Label of the $r_1m_0(d_2-1)(d_3-1)$ edges incident to the vertex having the label i in the level 2 having the other end vertex in the level 3 having the label $k-(i+1)$	Label of the $r_1m_0(d_2-1)(d_3-1)$ edges incident to the vertex having the label i in the level 2 having the other end vertex in the level 3 having the label $k-(i+2)$
$i=0$	$k-1$	$k-2$
$i=1$	$k-3$	$k-4$
$i=2$	$k-5$	$k-6$
$\vdots$	$\vdots$	$\vdots$
$i=\frac{k-4}{2}$	3	2
$i=\frac{k-2}{2}$	1	0
$i=\frac{k}{2}$	1	2
$\vdots$	$\vdots$	$\vdots$
$i=k-4$	$k-7$	$k-6$
$i=k-3$	$k-5$	$k-4$
$i=k-2$	$k-3$	$k-2$

Table 8: Edge labels of the edges incident to the vertices having the labels $k-1$ of the level 2 and having the other end in the level 3 corresponding to the case where k is even

Label i of a vertex in the level 2	Label of the $r_1m_0(d_2-1)(d_3-1)$ edges incident to the vertex having the label $k-1$ in the level 2 having the other end vertex in the level 3 having the label 0	Label of the $r_1m_0(d_2-1)(d_3-1)$ edges incident to the vertex having the label $k-1$ in the level 3 having the other end vertex in the level 3 having the label $k-1$
$i=k-1$	$k-1$	0

Table 9: Edge labels of the edges incident to the vertices having the labels i of the level 2 and having the other end in the level 3 corresponding to the case where k is odd

Label i of a vertex in the level 2	Label of the $r_1 m_0(d_2-1)(d_3-1)$ edges incident to the vertex having the label i in the level 2 having the other end vertex in the level 3 having the label $k-(i+1)$	Label of the $r_1 m_0(d_2-1)(d_3-1)$ edges incident to the vertex having the label i in the level 2 having the other end vertex in the level 3 having the label $k-(i+2)$
$i=0$	$k-1$	$k-2$
$i=1$	$k-3$	$k-4$
$i=2$	$k-5$	$k-6$
$\vdots$	$\vdots$	$\vdots$
$i=\frac{k-3}{2}$	2	1
$i=\frac{k-1}{2}$	0	1
$i=\frac{k+1}{2}$	2	3
$\vdots$	$\vdots$	$\vdots$
$i=k-4$	$k-7$	$k-6$
$i=k-3$	$k-5$	$k-4$
$i=k-2$	$k-3$	$k-2$

Table 10: Edge labels of the edges incident to the vertices having the labels $k-1$ of the level 2 and having the other end in the level 3 corresponding to the case where k is odd

Label of a vertex in the level 2	Label of the $r_1 m_0(d_2-1)(d_3-1)$ edges incident to the vertex having the label $k-1$ in the level 2 having the other end vertex in the level 3 having the label 0	Label of the $r_1 m_0(d_2-1)(d_3-1)$ edges incident to the vertex having the label $k-1$ in the level 3 having the other end vertex in the level 3 having the label $k-1$
$i=k-1$	$k-1$	0

From Tables 7, 8, 9, and 10, we have

$$|E_2(i)| = |E_2(j)| = 2r_1 m_0(d_2-1), \text{ for every pair } (i,j), 0 \le i < j \le k-1.$$

In general for the vertices in the level ℓ, for each ℓ, $4 \leq \ell \leq n$, we follow the Labeling Scheme 1. Then in each level ℓ, $4 \leq \ell \leq n$, for every pair (i, j), $0 \leq i < j \leq k-1$, we have

$$|V_\ell(i)| = |V_\ell(j)| = 2r_1 m_0 \prod_{i=2}^{\ell-1}(d_i - 1)$$

and for each ℓ, $3 \leq \ell \leq n-1$, for every pair (i, j), $0 \leq i < j \leq k-1$, we have

$$|E_\ell(i)| = |E_\ell(j)| = 2r_1 m_0 \prod_{i=2}^{\ell}(d_i - 1).$$

From Steps 1, 2, 3 and 4, we have the following facts:

1. In the level 0 the root is labeled with 0.
2. In the level 1, we have

$$|V_1(i)| = |V_1(j)| = m_0, \text{ for every pair } (i, j), 0 \leq i < j \leq k-1.$$

3. In the level 2, we have

$$|V_2(i)| = |V_2(j)| = 2r_1 m_0, \text{ for every pair } (i, j), 0 \leq i < j \leq k-1.$$

4. For every level ℓ, $3 \leq \ell \leq n$, we have

$$|V_\ell(i)| = |V_\ell(j)| = 2r_1 m_0 \prod_{i=2}^{\ell-1}(d_i - 1), \text{ for every pair } (i, j), 0 \leq i < j \leq k-1.$$

5. (i) $|E_0(i)| = |E_0(j)| = m_0$, for every pair (i, j), $0 \leq i < j \leq k-1$.

(ii) $|E_\ell(i)| = |E_\ell(j)| = 2r_1 m_0$, for every pair (i, j), $0 \leq i < j \leq k-1$.

(iii) For each ℓ, $2 \leq \ell \leq k-1$, we have $|E_\ell(i)| = |E_\ell(j)| = 2r_1 m_0 \prod_{i=2}^{\ell}(d_i - 1)$, for every pair (i, j), $0 \leq i < j \leq k-1$.

From the facts 1, 2, 3 and 4, we have

$$|V_T(i)| = |V_T(j)|, \text{ for every pair } (i, j), 1 \leq i < j \leq k-1 \tag{1}$$

and

$$|V_T(0)| - |V_T(i)| = 1, \text{ for every } i, 1 \leq i \leq k-1 \tag{2}$$

From the fact 5, we have

$$|E_T(i)| = |E_T(j)|, \text{ for every pair } (i,j), 0 \le i < j \le k-1. \tag{3}$$

The equations (1), (2) and (3) imply that T is k-equitable. ∎

Illustration

We give below in Figure 4 the 4-equitable labeling to the first two levels degree specific symmetric tree, where the 4-equitable labeling defined here in this example is given as in the proof of Theorem 2.2.

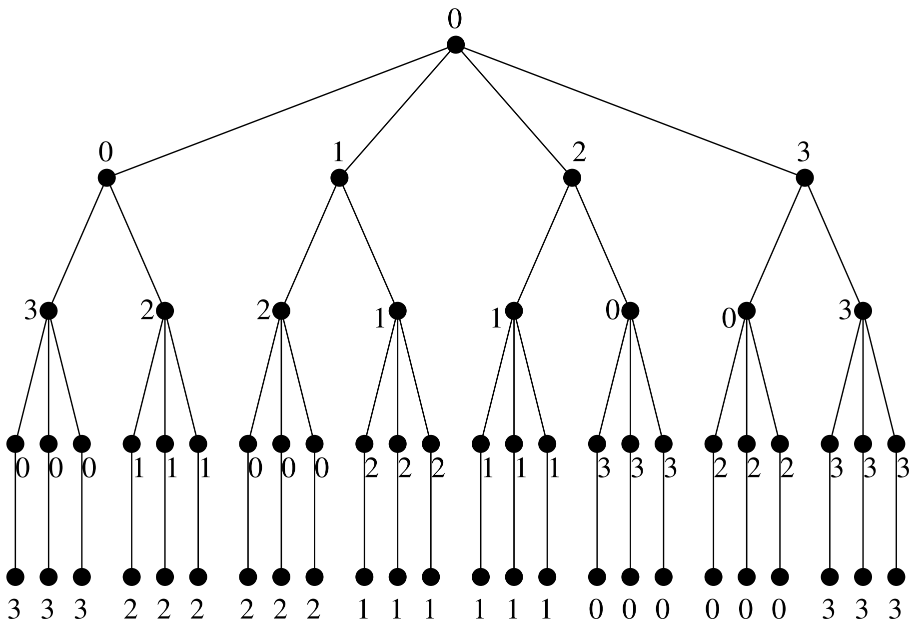

Figure 4: The 4-equitable labeling of the first two levels degree specific symmetric tree

References

[1] I. Cahit, Cordial graphs : a weaker version of graceful and harmonious graphs, Ars Combin., 23 (1987), 201–208.

[2] I. Cahit, Status of graceful tree conjecture in 1989, Topics in Combinatorics and Graph Theory, R. Bodendiek and R. Henn (eds.), Phsica-Verlag, Heidelberg, 1990. http://dx.doi.org/10.1007/978-3-642-46908-4-20.

[3] I. Cahit, Recent results and open problems on cordial graphs, Contemporary methods in graph theory, R. Bodendiek, Wissenschaftsverlag, Mannheim, 1990, 209–230.

[4] S.W. Golomb, How to number a graph, Graph Theory and Computing, R.C. Reed, Ed., Academic Press, New York, 1972, 23–37. http://dx.doi.org/10.1016/b978-1-4832-3187-8.50008-8.

[5] J.A. Gallian, A dynamic survey of graph labeling, Electronic Journal of Combinatorics, 19 (2013).

[6] A. Kotzig, Decompositions of a complete graph into $4k$-gons (in Russia), Matematicky Casopis, 15 (1965), 229–233.

[7] G. Ringel, Problem 25, in Theory of Graphs and its Applications, Proceedings of the Symposium Smolenice (1963), Prague (1964), 162.

[8] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs, International Symposium, Rome, July 1966, Dunod, Paris, 1967, 349–355.

[9] Z. Szaniszlo, k-equitable labelings of cycles and some other graphs, Ars. Combin., 37 (1994), 49–63.

[10] D.E. Speyer and Z. Szaniszlo, Every tree is 3-equitable, Discrete Mathematics, 220 (2000), 283–289.