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The Laplacian Energy of The Sum Geometric Arithmetic Means of a Graph

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Abstract

Let G be a graph with n vertices and m edges. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix of G, and let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the Laplacian matrix of G. An earlier much studied quantity $E(G) = \sum_{i=1}^{n} |\lambda_i|$ is the energy of the graph G.
In this paper, we introduce the concept of the Laplacian matrix of the sum geometric arithmetic means of a graph G, denoted by $L_{SGAM}(G)$, and the concept of the Laplacian energy of the sum geometric arithmetic means of a graph G, denoted by $LE_{SGAM}(G)$. Also, we find the characteristic polynomial of the Laplacian matrix of the sum geometric arithmetic means of some standard graphs . Finally, we compute the Laplacian energy of the sum geometric arithmetic means of few families of graphs.

Key words: The Laplacian matrix of the sum geometric arithmetic means of a graph G.

2010 Mathematics Subject Classification : 05C50

1 Introduction

In this paper we are concerned with simple graphs. Let G be such a graph, possessing n vertices and m edges. Let $deg(i)$ be the degree of the i^{th} vertex of $G, i = 1, 2, \ldots, n$. The spectrum of the graph G, consisting of the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, is the spectrum of its adjacency matrix [3]. The Laplacian spectrum of the graph G, consisting of the

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numbers $\mu_1, \mu_2, \ldots, \mu_n$, is the spectrum of its Laplacian matrix. The energy of a graph G was defined by I. Gutman in 1978 as the sum of the absolute values of eigenvalues of G [10]. The concept of graph energy has origin in chemistry which is used to estimate the total π-electron energy of a molecule. In chemistry the conjugated hydrocarbons can be represented by a graph called molecular graph whose eigenvalues with respect to adjacency matrix $A(G)$ represent the energy level of the electron in the molecule. In Huckle theory the sum of the energies of all the electrons in a molecule is called the π - electron energy of a molecule [16]. In spectral graph theory, the energy - like quantities such as Laplacian energy, distance energy, color energy, color Laplacian energy of a graph etc., are studied in [1, 10, 11, 3].

The concept of color energy was introduced by C. Adiga et al. in [1]., P. G. Bhat and S. D'Souza [3] have studied the color Laplacian energy of a graph. Let G be a colored graph on n vertices and m edges. The color Laplacian matrix of G is defined as $Lc(G) = D(G) - Ac(G)$ where $D(G) = diag\Big(deg(v_1), deg(v_2), \ldots, deg(v_n)\Big)$ represents the diagonal matrix with vertex degrees $deg(v_1), deg(v_2), \ldots, deg(v_n)$ of $v_1, v_2, \ldots, v_n$ of G and $Ac(G)$, the color matrix. The eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$ of $Lc(G)$ are called color Laplacian eigenvalues of the graph G. If auxiliary color eigenvalues $yi, \quad i = 1, 2, 3, \cdots, n$ are defined as $yi = \mu_i - 2nm$, then color Laplacian energy of G is defined as $\sum_{i=1}^{n} |\gamma_i|$. Laplacian matrix $L = L(G)$ of (n, m) graph is defined via its matrix elements as [10],

$$l_{ij} = \begin{cases} -1, & \text{if } i \neq j \quad v_i \text{ and } v_j \text{ are adjcent,} \\ 0, & \text{if } i \neq j \quad v_i \text{ and } v_j \text{ are not adjcent, ,} \\ deg(i) & \text{if i} = j, \end{cases}$$

and its Laplacian eigenvalues are $\mu_1, \mu_2, \ldots, \mu_n$, then the Laplacian energy of the graph G is defined as [10]

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|.$$

In 2013, Aouchiche and Hansen [2] introduced the distance signless Laplacian matrix of a connected graph G as the $n \times n$ matrix defined by $DQ(G) = Tr(G) + D_G$, where D_G is the distance matrix of G and $Tr(G)$ is the diagonal matrix of vertex transmissions of G. In 2015, Sehgehalli et al. [18] defined the arithmetic - geometric adjacency matrix (AG matrix) of G, denoted by $A_{ag} = (g_{ij})$, where $g_{ij} = \frac{deg(v_i)+deg(v_j)}{2\sqrt{deg(v_i)deg(v_j)}}$ if $v_iv_j \in E(G)$ and 0 otherwise. Also in 2015, E. Sampathkumar, S. V. Roopa, K. A. Vidya and M. A. Sriraj

[15] have introduced the partition energy of a graph $E_{P_k}(G)$ and computed partition energy of some families of graphs with respect to a given partition.

In 2016, V. Kaladevi and A. Abinayaa [12] computed the detour distance laplacian energy of standard graphs. In 2017, B. Sharada, M. I. Sowaity and I. Gutman, [17] introduced the Laplacian sum - eccentricity matrix LS_e of a graph G, and its Laplacian sum - eccentricity energy $LS_eE = \sum_{i=1}^{n} |\eta_i|$, where $\eta_i = \zeta_i - 2nm$ and where $\zeta_1; \zeta_2; \ldots; \zeta_n$ are the eigenvalues of LS_e. And In 2017, P. G. Bhat and S. D'Souza [4] introduced the new concept of color Signless Laplacian energy $LE_c^+(G)$. It depends on the underlying graph G and the colors of the vertices. In 2017, D. Nilanjan investigated the eccentricity version of Laplacian energy of a graph G. In 2019, Xin Guo and Yubin Gao [8] obtained some lower and upper bounds on arithmetic - geometric radius and arithmetic - geometric energy. Also in 2019, E. Sampathkumar, S. V. Roopa, K. A. Vidya and M. A. Sriraj [16] have introduced the concept of partition laplacian energy of a graph $lE_{P_k}(G)$ and computed partition energy of some families of graphs with respect to a given partition. In 2022, M. M. Mohsen and S. S. Mahde [14] have introduced the concept of the sum geometric arithmetic means matrix of a graph G, denoted by $A_{SGAM}(G)$ and defined as follows

$$A_{SGAM}(G) = (a_{ij});$$

$$a_{ij} = \begin{cases} \sqrt{deg(u)deg(v)} + \frac{deg(u)+deg(v)}{2}, & \text{if the vertex } \quad u \quad \text{ is adjacent to the vertex v} \\ 0, & \text{otherwise,} \end{cases}$$

where $deg(u), \quad deg(v)$ is degree of the vertex u and v, respectively.
Motivated by recent work on energy of a graph, in this paper we introduce the Laplacian matrix of the sum geometric arithmetic means of a graph G.
We begin with basic definitions needed in this paper.

Definition 1.1. [6] A complete graph is a simple graph in which any two vertices are adjacent and denoted by K_n.

Definition 1.2. [5] A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = p$ and $|Y| = q$, such a graph is denoted by $K_{p,q}$.
The complete bipartite graph $K_{p,q}$ has $p + q$ vertices and pq edges.

Definition 1.3. [7, 13] The crown graph S_p^0 for an integer $p \geq 2$ is the graph with vertex set $\{u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p\}$ and edge set $\{u_i v_j : 1 \leq i, j \leq p, i \neq j\}$. S_p^0 is therefore equivalent to the complete bipartite graph $K_{p,p}$ with horizontal edges removed.

Definition 1.4. [6] A star graph is a complete bipartite graph $K_{p,q}$ with $|p| = 1$ or $|q| = 1$, denoted by $K_{1,n-1}$ i.e. A star graph is simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if $|X| = 1$ and $|Y| = n-1$, such a graph is denoted by $K_{1,n-1}$.

2 Main Results

The Laplacian matrix of a graph and it's eigenvalues can be used in several areas of mathematical research and have various applications in physical and chemical theories.

Definition 2.1. The laplacian of the sum geometric arithmetic means matrix denoted by $L_{SGAM}(G)$ and defined as

$$L_{SGAM}(G) = D(G) - A_{SGAM}(G),$$

where $D(G)$ is the degree matrix of a graph G and $A_{SGAM}(G)$ is the sum geometric arithmetic means matrix of a graph G.

Definition 2.2. The laplacian energy of sum geometric arithmetic means denoted by $LE_{SGAM}(G)$ and defined as

$$LE_{SGAM}(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|,$$

where $\mu_i \quad (i = 1, 2, \ldots, n)$ are the eigenvalues of $L_{SGAM}(G)$, m is the the number of edges of G and n is the number of vertices of G.

Theorem 2.3. For the complete graph K_n, the Laplacian energy of the sum geometric arithmetic means is $LE_{SGAM}(K_n) = 4(n-1)^2$.

Proof: Since

$$A_{SGAM}(K_n)=\begin{pmatrix} 0 & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 2(n-1) & 2(n-1)\\ 2(n-1) & 0 & 2(n-1) & \dots & 2(n-1) & 2(n-1) & 2(n-1)\\ 2(n-1) & 2(n-1) & 0 & \dots & 2(n-1) & 2(n-1) & 2(n-1)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 0 & 2(n-1) & 2(n-1)\\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 0 & 2(n-1)\\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 2(n-1) & 0 \end{pmatrix}_{n\times n},$$

$$D(K_n)=\begin{pmatrix} n-1 & 0 & 0 & \dots & 0 & 0 & 0\\ 0 & n-1 & 0 & \dots & 0 & 0 & 0\\ 0 & 0 & n-1 & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & n-1 & 0 & 0\\ 0 & 0 & 0 & \dots & 0 & n-1 & 0\\ 0 & 0 & 0 & \dots & 0 & 0 & n-1 \end{pmatrix}_{n\times n},$$

then

$$L_{SGAM}(K_n)=D(K_n)-A_{SGAM}(K_n)$$

$$=\begin{pmatrix} n-1 & -2(n-1) & -2(n-1) & \dots & -2(n-1) & -2(n-1) & -2(n-1)\\ -2(n-1) & n-1 & -2(n-1) & \dots & -2(n-1) & -2(n-1) & -2(n-1)\\ -2(n-1) & -2(n-1) & n-1 & \dots & -2(n-1) & -2(n-1) & -2(n-1)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ -2(n-1) & -2(n-1) & -2(n-1) & \dots & n-1 & -2(n-1) & -2(n-1)\\ -2(n-1) & -2(n-1) & -2(n-1) & \dots & -2(n-1) & n-1 & -2(n-1)\\ -2(n-1) & -2(n-1) & -2(n-1) & \dots & -2(n-1) & -2(n-1) & n-1 \end{pmatrix}_{n\times n},$$

to find the characteristic polynomial, we solve $det(\mu I_n - L_{SGAM}(K_n))=0$,
so $det\Big(\mu I_n - L_{SGAM}(K_n)\Big)$

$$=\begin{vmatrix} \mu-(n-1) & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 2(n-1) & 2(n-1)\\ 2(n-1) & \mu-(n-1) & 2(n-1) & \dots & 2(n-1) & 2(n-1) & 2(n-1)\\ 2(n-1) & 2(n-1) & \mu-(n-1) & \dots & 2(n-1) & 2(n-1) & 2(n-1)\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & \mu-(n-1) & 2(n-1) & 2(n-1)\\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 2(n-1) & \mu-(n-1) & 2(n-1)\\ 2(n-1) & 2(n-1) & 2(n-1) & \dots & 2(n-1) & 2(n-1) & \mu-(n-1) \end{vmatrix}_{n\times n},$$

then the characteristic polynomial is

$$P\Big(L_{SGAM}(K_n),\mu\Big) = \Big(\mu + (n-1)(2n-3)\Big)\Big(\mu - 3(n-1)\Big)^{n-1},$$

hence the eigenvalues are $\mu_1 = -(n-1)(2n-3)$, $\mu_2 = \mu_3 = \mu_4 = \cdots = \mu_n = 3(n-1)$, thus the laplacian energy of the sum geometric arithmetic means is

$$\begin{aligned} LE_{SGAM}(K_n) &= \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| = \left|-(n-1)(2n-3) - \frac{n(n-1)}{n}\right| + \sum_{i=1}^{n-1}\left|3(n-1) - \frac{n(n-1)}{n}\right| \\ &= \Big|-(n-1)(2n-3) - (n-1)\Big| + \sum_{i=1}^{n-1}\Big|3(n-1) - (n-1)\Big| \\ &= |-(n-1)(2n-3) - (n-1)| + \sum_{i=1}^{n-1}|2(n-1)| \\ &= (n-1)(2n-2) + (n-1) \times 2(n-1) \\ &= (n-1) \times 2(n-1) + (n-1) \times 2(n-1) = 4(n-1)^2. \end{aligned}$$

∎

Theorem 2.4. For the complete bipartite graph $K_{p,p}$, the Laplacian energy of the sum geometric arithmetic means is $LE_{SGAM}(K_{p,p}) = 4p(2p-1)$.

Proof: From definition of $A_{SGAM}(G)$, we have

$$A_{SGAM}(K_{p,p}) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 2p & 2p & 2p & \dots & 2p \\ 0 & 0 & 0 & \dots & 0 & 2p & 2p & 2p & \dots & 2p \\ 0 & 0 & 0 & \dots & 0 & 2p & 2p & 2p & \dots & 2p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 2p & 2p & 2p & \dots & 2p \\ 2p & 2p & 2p & \dots & 2p & 0 & 0 & \dots & 0 & 0 \\ 2p & 2p & 2p & \dots & 2p & 0 & 0 & \dots & 0 & 0 \\ 2p & 2p & 2p & \dots & 2p & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2p & 2p & 2p & \dots & 2p & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{2p\times 2p},$$

$$D(K_{p,p}) = \begin{pmatrix} p & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & p & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & p & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & p & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & p & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & p & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & p & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & p \end{pmatrix}_{2p\times 2p},$$

since $L_{SGAM}(K_{p,p}) = D(K_{p,p}) - A_{SGAM}(K_{p,p})$, therefore $L_{SGAM}(K_{p,p}) =$

$$\begin{pmatrix} p & 0 & 0 & \dots & 0 & -2p & -2p & -2p & \dots & -2p \\ 0 & p & 0 & \dots & 0 & -2p & -2p & -2p & \dots & -2(p \\ 0 & 0 & p & \dots & 0 & -2p & -2p & -2p & \dots & -2p \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & p & -2p & -2p & -2p & \dots & -2p \\ -2p & -2p & -2p & \dots & -2p & p & 0 & 0 & \dots & 0 \\ -2p & -2p & -2p & \dots & -2p & 0 & p & 0 & \dots & 0 \\ -2p & -2p & -2p & \dots & -2p & 0 & 0 & p & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -2p & -2p & -2p & \dots & -2p & 0 & 0 & \dots & 0 & p \end{pmatrix}_{2p\times 2p}.$$

To find the characteristic polynomial, we solve $det(\mu I_{2p}-L_{SGAM}(K_{p,p})=0$, so $det\Big(\mu I_{2p}-L_{SGAM}(K_{p,p})\Big)$

$$= \begin{vmatrix} \mu-p & 0 & 0 & \dots & 0 & 0 & 2p & 2p & \dots & 2p \\ 0 & \mu-p & 0 & \dots & 0 & 2p & 2p & 2p & \dots & 2p \\ 0 & 0 & \mu-p & \dots & 0 & 2p & 2p & \dots & 2p & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & \mu-p & 2p & 2p & 2p & \dots & 2p \\ 2p & 2p & 2p & \dots & 2p & \mu-p & 0 & 0 & \dots & 0 \\ 2p & 2p & 2p & \dots & 2p & 2p & \mu-p & 0 & \dots & 0 \\ 2p & 2p & 2p & \dots & 2p & 0 & 0 & \mu-p & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2p & 2p & 2p & \dots & 2p & 0 & 0 & 0 & \dots & \mu-p \end{vmatrix},$$

so the characteristic polynomial is

$$P(L_{SGAM}(K_{p,p}),\mu) = (\mu+p)^{2p-2}\Big(\mu+p(2p+1)\Big)\Big(\mu-p(2p-1)\Big),$$

hence the eigenvalues are $\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_{2p-2} = -p$, $\mu_{2p-1} = -p(2p+1)$, $\mu_{2p} = p(2p-1)$, thus the laplacian energy of the sum geometric arithmetic means is

$$\begin{aligned} LE_{SGAM}(K_{p,p}) &= \sum_{i=1}^{2p}\left|\mu_i - \frac{2m}{n}\right| = \sum_{i=1}^{2p-2}\left|\mu_i - \frac{2m}{n}\right| + \left|\mu_{2p-1} - \frac{2m}{n}\right| + \left|\mu_{2p} - \frac{2m}{n}\right| \\ &= \sum_{i=1}^{2p-2}\left|-p - \frac{2p^2}{2p}\right| + \left|-p(2p+1) - \frac{2p^2}{2p}\right| + \left|p(2p-1) - \frac{2p^2}{2p}\right| \\ &= \sum_{i=1}^{2p-2}\left|p + \frac{2p^2}{2p}\right| + \left|p(2p+1) + \frac{2p^2}{2p}\right| + \left|p(2p+1) + \frac{2p^2}{2p}\right| + |p(2p-1) - p| \\ &= \sum_{i=1}^{2p-2}|p+p| + |p(2p+1)+p| + |p(2p+1)+p| + \Big|p(2p-1) - p\Big| \\ &= (2p-2)(p+p) + 2p(p+1) + 2p(p-1) = 4p(p-1) + 2p(p+1+p-1) \\ &= 4p(p-1) + 4p^2 = 4p(p-1+p) = 4p(2p-1). \end{aligned}$$

■

Theorem 2.5. For the star graph $K_{1,n-1}$, the Laplacian energy of the sum geometric arithmetic means is $L_{SGAM}(K_{1,n-1}) = (n-2) + \frac{2(n-1)(n-2)}{n} + \sqrt{n^2 + 4(-1)^{-(n+1)} + \prod_{i=1}^{n}\mu_i}$.

Proof: $A_{SGAM}(K_{1,n-1})=$

$$\begin{pmatrix} 0 & \sqrt{n-1}+\frac{n}{2} & \sqrt{n-1}+\frac{n}{2} & \dots & \sqrt{n-1}+\frac{n}{2} & \sqrt{n-1}+\frac{n}{2} & \sqrt{n-1}+\frac{n}{2} \\ \sqrt{n-1}+\frac{n}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ \sqrt{n-1}+\frac{n}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sqrt{n-1}+\frac{n}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ \sqrt{n-1}+\frac{n}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ \sqrt{n-1}+\frac{n}{2} & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}_{n\times n},$$

$$D(K_{1,n-1}) = \begin{pmatrix} (n-1) & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}_{n\times n},$$

since $L_{SGAM}(K_{1,n-1}) = D(K_{1,n-1}) - A_{SGAM}(K_{1,n-1})$, therefore $L_{SGAM}(K_{1,n-1}) =$

$$\begin{pmatrix} (n-1) & -\sqrt{n-1}-\frac{n}{2} & -\sqrt{n-1}-\frac{n}{2} & \dots & -\sqrt{n-1}-\frac{n}{2} & -\sqrt{n-1}-\frac{n}{2} & -\sqrt{n-1}-\frac{n}{2} \\ -\sqrt{n-1}-\frac{n}{2} & 1 & 0 & \dots & 0 & 0 & 0 \\ -\sqrt{n-1}-\frac{n}{2} & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\sqrt{n-1}-\frac{n}{2} & 0 & 0 & \dots & 1 & 0 & 0 \\ -\sqrt{n-1}-\frac{n}{2} & 0 & 0 & \dots & 0 & 1 & 0 \\ -\sqrt{n-1}-\frac{n}{2} & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}_{n\times n}.$$

To find the characteristic polynomial, we solve $det(\mu I_n - L_{SGAM}(K_{1,n-1})) = 0$,
so $det(\mu I_n - L_{SGAM}(K_{1,n-1})) =$

$$\begin{vmatrix} \mu-(n-1) & \sqrt{n-1}+\frac{n}{2} & \sqrt{n-1}+\frac{n}{2} & \dots & \sqrt{n-1}+\frac{n}{2} & \sqrt{n-1}+\frac{n}{2} & \sqrt{n-1}+\frac{n}{2} \\ \sqrt{n-1}+\frac{n}{2} & \mu-1 & 0 & \dots & 0 & 0 & 0 \\ \sqrt{n-1}+\frac{n}{2} & 0 & \mu-1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sqrt{n-1}+\frac{n}{2} & 0 & 0 & \dots & \mu-1 & 0 & 0 \\ \sqrt{n-1}+\frac{n}{2} & 0 & 0 & \dots & 0 & \mu-1 & 0 \\ \sqrt{n-1}+\frac{n}{2} & 0 & 0 & \dots & 0 & 0 & \mu-1 \end{vmatrix},$$

then the characteristic polynomial is

$$P\Big(L_{SGAM}(K_{1,n-1}), \mu\Big) = (\mu+1)^{n-2}\left(\mu^2 + n\mu + (-1)^n \prod_{i=1}^{n} \mu_i\right),$$

hence the eigenvalues are $\mu_1 = \mu_2 = \mu_3 = \dots = \mu_{n-2} = -1$, $\mu_{n-1} = \frac{-n+\sqrt{n^2+4(-1)^{-(n+1)}+\prod_{i=1}^{n}\mu_i}}{2}$, $\mu_n = \frac{-n-\sqrt{n^2+4(-1)^{-(n+1)}+\prod_{i=1}^{n}\mu_i}}{2}$, thus the laplacian energy of the sum geometric arithmetic means is

$$\begin{aligned} LE_{SGAM}(K_{1,n-1}) &= \sum_{i=1}^{n}\left|\mu_i - \frac{2m}{n}\right| \\ &= \sum_{i=1}^{n-2}\left|\mu_i - \frac{2(n-1)}{n}\right| + \left|\mu_{n-1} - \frac{2m}{n}\right| + \left|\mu_n - \frac{2m}{n}\right| \\ &= \sum_{i=1}^{n-2}\left|-1 - \frac{2m}{n}\right| + \left|\frac{-n+\sqrt{n^2+4(-1)^{-(n+1)}+\prod_{i=1}^{n}\mu_i}}{2} - \frac{2(n-1)}{n}\right| \\ &+ \left|\frac{-n-\sqrt{n^2+4(-1)^{-(n+1)}+\prod_{i=1}^{n}\mu_i}}{2} - \frac{2(n-1)}{n}\right| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-2}\left|1+\frac{2(n-1)}{n}\right| + \frac{-n+\sqrt{n^2+4(-1)^{-(n+1)}+\prod_{i=1}^{n}\mu_i}}{2} - \frac{2(n-1)}{n} \\
&+ \frac{n+\sqrt{n^2+4(-1)^{-(n+1)}+\prod_{i=1}^{n}\mu_i}}{2} + \frac{2(n-1)}{n} \\
&= (n-2) + \frac{2(n-1)(n-2)}{n} + \sqrt{n^2+4(-1)^{-(n+1)}+\prod_{i=1}^{n}\mu_i}.
\end{aligned}
$$

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Theorem 2.6. For the crown graph S_p^0, the laplacian energy of the sum geometric arithmetic means is $LE_{SGAM}(S_p^0) = 4(p-1)^2$.

Proof: Applying the definition of $A_{SGAM}(G)$ on S_p^0, we obtain $A_{SGAM}(S_p^0)$=

$$
\begin{pmatrix}
0 & 0 & 0 & \dots & 0 & 0 & 2(p-1) & 2(p-1) & \dots & 2(p-1) \\
0 & 0 & 0 & \dots & 0 & 2(p-1) & 0 & 2(p-1) & \dots & 2(p-1) \\
0 & 0 & 0 & \dots & 0 & 2(p-1) & 2(p-1) & 0 & \dots & 2(p-1) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & \dots & 0 & 2(p-1) & 2(p-1) & 2(p-1) & \dots & 0 \\
0 & 2(p-1) & 2(p-1) & \dots & 2(p-1) & 0 & 0 & \dots & 0 & 0 \\
2(p-1) & 0 & 2(p-1) & \dots & 2(p-1) & 0 & 0 & 0 & \dots & 0 \\
2(p-1) & 2(p-1) & 0 & \dots & 2(p-1) & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
2(p-1) & 2(p-1) & 2(p-1) & \dots & 0 & 0 & 0 & \dots & 0 & 0
\end{pmatrix}_{2p\times 2p}.
$$

$$
D(S_p^0) = \begin{pmatrix}
p-1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & p-1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & p-1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & \dots & p-1 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & p-1 & 0 & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & p-1 & 0 & \dots & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & p-1 \dots & 0 & \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & p-1
\end{pmatrix}_{2p\times 2p},
$$

since $L_{SGAM}(S_p^0) = D(S_p^0) - A_{SGAM}(S_p^0)$, therefore $L_{SGAM}(S_p^0) =$

$$\begin{pmatrix}
p-1 & 0 & 0 & \dots & 0 & 0 & -2(p-1) & -2(p-1) & \dots & -2(p-1) \\
0 & p-1 & 0 & \dots & 0 & -2(p-1) & 0 & -2(p-1) & \dots & -2(p-1) \\
0 & 0 & p-1 & \dots & 0 & -2(p-1) & -2(p-1) & 0 & \dots & -2(p-1) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\
0 & 0 & 0 & \dots & p-1 & -2(p-1) & -2(p-1) & -2(p-1) & \dots & 0 \\
0 & -2(p-1) & -2(p-1) & \dots & -2(p-1) & 0 & p-1 & 0 & \dots & 0 \\
-2(p-1) & 0 & -2(p-1) & \dots & -2(p-1) & 0 & p-1 & 0 & \dots & 0 \\
-2(p-1) & -2(p-1) & 0 & \dots & -2(p-1) & 0 & 0 & p-1 & \dots & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-2(p-1) & -2(p-1) & -2(p-1) & \dots & 0 & 0 & 0 & \dots & 0 & p-1
\end{pmatrix}_{2p\times 2p}.$$

To find the characteristic polynomial, we solve $det(\mu I_{2p} - L_{SGAM}(S_p^0)) = 0$,
so $det(\mu I_{2p} - L_{SGAM}(S_p^0)) =$

$$\begin{vmatrix}
\mu-(p-1) & 0 & 0 & \dots & 0 & 0 & 2(p-1) & 2(p-1) & \dots & 2(p-1) \\
0 & \mu-(p-1) & 0 & \dots & 0 & 2(p-1) & 0 & 2(p-1) & \dots & 2(p-1) \\
0 & 0 & \mu-(p-1) & \dots & 0 & 2(p-1) & 2(p-1) & 0 & \dots & 2(p-1) \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \mu-(p-1) & 2(p-1) & 2(p-1) & 2(p-1) & \dots & 0 \\
0 & 2(p-1) & 2(p-1) & \dots & 2(p-1) & 0 & \mu-(p-1) & 0 & \dots & 0 \\
2(p-1) & 0 & 2(p-1) & \dots & 2(p-1) & 0 & \mu-(p-1) & 0 & \dots & 0 \\
2(p-1) & 2(p-1) & 0 & \dots & 2(p-1) & 0 & 0 & \mu-(p-1) & \dots & 0 \\
\vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
2(p-1) & 2(p-1) & 2(p-1) & \dots & 0 & 0 & 0 & \dots & 0 & \mu-(p-1)
\end{vmatrix}_{2p\times 2p},$$

then the characteristic polynomial is

$$P\Big(L_{SGAM}(S_p^0), \mu\Big) = \Big(\mu+(p-1)(2p-3)\Big)\Big(\mu-3(p-1)\Big)^{p-1}\Big(\mu-(p-1)(2p-1)\Big)\Big(\mu+(p-1)\Big)^{p-1},$$

hence the eigenvalues are $\mu_1 = -(p-1)(2p-3)$, $\mu_2 = (p-1)(2p-1)$, $\mu_3 = \mu_4 = \cdots = \mu_p = \mu_{p+1} = \mu_{p+2} = 3(p-1)$, $\mu_{p+3} = \mu_{p+4} = \cdots = \mu_{2p-1} = \mu_{2p} = -(p-1)$, thus the laplacian energy of the sum geometric arithmetic means is

$$\begin{aligned}
LE_{SGAM}(S_p^0) &= \sum_{i=1}^{2p}\left|\mu_i - \frac{2m}{n}\right|\left|-(p-1)(2p-3) - \frac{p(p-1)}{p}\right| + \sum_{i=1}^{p-1}\left|3(p-1) - \frac{p(p-1)}{p}\right| \\
&= |-(p-1)(2p-3) - (p-1)| + (p-1)\times 2(p-1) \\
&= (p-1)(2p-2) + (p-1)\times 2(p-1) = 2(p-1)^2 + 2(p-1)^2 = 4(p-1)^2.
\end{aligned}$$

■

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