International Journal of Applied Graph Theory
Vol.2, No.1 (2018), 01 - 06.
$ISSN(Online) : 2456 - 7884$

Total Resolving Number of Graphs - Some Characterizations

J. Paulraj Joseph and N. Shunmugapriya

Department of Mathematics
Manonmaniam Sundaranar University
Tirunelveli - 627 012
prof.jpaulraj@gmail.com , nshunmugapriya2013@gmail.com

Abstract

Let $G = (V, E)$ be a simple connected graph of order $n \geq 4$. An ordered subset W of V is said to be a resolving set of G if every vertex is uniquely determined by its vector of distances to the vertices in W. The minimum cardinality of a resolving set is called the resolving number of G and is denoted by $r(G)$. As an extension, the total resolving number was introduced in [5] as the minimum cardinality taken over all resolving sets in which $\langle W \rangle$ has no isolates and it is denoted by $tr(G)$. In this paper, we characterize 1-connected graphs for which $tr(G) = n - 2$ and bipartite graphs for which $tr(G) = 2$.

Key words: Resolving number, Total resolving number

2010 Mathematics Subject Classification : Primary 05C12, Secondary 05C35

1 Introduction

Let $G = (V, E)$ be a finite, simple, connected and undirected graph. The *degree* of a vertex v in a graph G is the number of edges incident to v and it is denoted by $d(v)$. The maximum degree in a graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest u-v path in G. The maximum value of distance between vertices of G is called its *diameter*. P_n denote the *path* on n vertices. C_n denote the *cycle* on n vertices. K_n denote the *complete graph* on n vertices. A graph is *acyclic* if it has no cycles. A *tree*

* Corresponding Author: *N. Shunmugapriya*
Ψ Received on August 31, 2017 / Revised on November 19, 2017 / Accepted on November 20, 2017

is a connected acyclic graph. A *complete bipartite graph* is denoted by $K_{s,t}$. A *star* is denoted by $K_{1,n-1}$. A *tree* obtained by joining the centres of two stars $K_{1,s}$ and $K_{1,t}$ by an edge is called a *bistar* and it is denoted by $B_{s,t}$. The *join* $G+H$ consists of $G \cup H$ and all edges joining a vertex of G and a vertex of H.

For a cut vertex v of a connected graph G, suppose that the disconnected graph $G \setminus \{v\}$ has k components $G_1, G_2, \ldots, G_k$ $(k \geq 2)$. The induced subgraphs $B_i = G[V(G_i) \cup \{v\}]$ are connected and referred to as the *brances* of G at v. To *identify* non adjacent vertices x and y of a graph G is to replace these vertices by a single vertex which is incident to all the edges which were incident in G to either x or y. For any two integers x and y, $x \sim y$ denotes the difference between x and y. For $l \geq 1$, $N_l(W) = \{v \in V \setminus W \ / \ d(v,W) = l\}$. A vertex of degree at least 3 in a graph G is called a *major vertex* of G. Any end vertex u of G is said to be a *terminal vertex* of a major vertex v of G if $d(u,v) < d(u,w)$ for every other major vertex w of G. The *terminal degree* $ter(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree.

If $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ is an ordered set, then the ordered k-tuple $(d(v,w_1), d(v,w_2), ..., d(v,w_k))$ is called the representation of v with respect to W and it is denoted by $r(v|W)$. Since the representation for each $w_i \in W$ contains exactly one 0 in the i^{th} position, all the vertices of W have distinct representations. W is called a *resolving set* for G if all the vertices of $V \setminus W$ also have distinct representations. The minimum cardinality of a resolving set is called the *resolving number* of G and it is denoted by $r(G)$. In [5], we introduced and studied total resolving number. If W is a resolving set and the induced subgraph $\langle W \rangle$ has no isolates, then W is called a *total resolving set* of G. The minimum cardinality taken over all total resolving sets of G is called the *total resolving number* of G and is denoted by $tr(G)$.

In this paper, we characterize 1-connected graphs for which $tr(G) = n-2$ and bipartite graphs for which $tr(G) = 2$.

2 Total Resolving Number of Graphs

The total resolving number of some well-known classes of graphs have been determined in [5]. The following results are used in the next section.

Observation 2.1. [5] Let G be a graph of order $n \geq 3$. Then $2 \leq tr(G) \leq n-1$.

Theorem 2.2. [5] Let G be a graph of order $n \geq 3$. Then $tr(G) = n-1$ if and only if $G \cong K_n$ or $K_{1,n-1}$.

Remark 2.3. [5] Let $W = \{w_1, w_2\}$ be a total resolving set of G and d be the diameter of G. Then $|N_l(W)| \leq 3$, $1 \leq l \leq d-1$.

3 Two Characterizations

In this section, we characterize 1-connected graphs for which $tr(G) = n-2$ and bipartite graphs for which $tr(G) = 2$.

Notation 3.1. If H and K are two graphs, then the graph obtained by identifying one center of H with one center of K is denoted by $H * K$ and the graph obtained by joining one center of H to one center of K is denoted by $H\ e\ K$.

Theorem 3.2. Let G be a 1-connected graph of order $n \geq 4$. Then $tr(G) = n-2$ if and only if G is isomorphic to $B_{s,t}$, $s \geq 1$, $t \geq 1$ or $2K_3 + e$ or $K_{1,s} * K_t$, $s \geq 1$, $t \geq 3$ or $K_3 * K_t$, $t \geq 3$ or $K_3\ e\ K_{1,s}$, $s \geq 3$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$.

Assume that $tr(G) = n-2$. Let W be a total resolving set of G.

Let v_1 be a cut vertex of G. Let $C_1, C_2, \ldots, C_r$, $r \geq 2$ be the components of $G \setminus \{v_1\}$. Then there are r branches at v_1. Let $B_1, B_2, \ldots, B_r$ be such branches. First, we claim that at most two branches contain a cycle. Suppose not. Let B_1, B_2, B_3 be such branches. By Observation 2.1, $tr(B_1) \leq |B_1| - 1$, $tr(B_2) \leq |B_2| - 1$ and $tr(B_3) \leq |B_3| - 1$. Then there exist $v_i \in V(B_1)$, $v_j \in V(B_2)$ and $v_k \in V(B_3)$, $i, j, k \neq 1$ such that $V(B_1) \setminus \{v_i\}$, $V(B_2) \setminus \{v_j\}$ and $V(B_3) \setminus \{v_k\}$ are total resolving sets of B_1, B_2 and B_3 respectively. Clearly, $V(G) \setminus \{v_i, v_j, v_k\}$ is a total resolving set of G with cardinality $n-3$, which is a contradiction.

Thus at most two branches contain a cycle. We consider the following three cases.

Case 1 : G is a tree.

First, we claim that G contains one or two exterior major vertices. Suppose that G contains more than two exterior major vertices. Let v_i, v_j and v_k be such vertices. Let v_i', v_j' and v_k' be the terminal vertices of v_i, v_j and v_k respectively. Then $V(G) \setminus \{v_i', v_j', v_k'\}$ is a total resolving set of G with cardinality $n-3$, which is a contradiction. Thus G has one or two exterior major vertices. If G has exactly one exterior major vertex, then we claim that $G \cong B_{1,t}$, $t \geq 3$. Clearly, v_1 is the exterior major vertex.

Now, we claim that exactly one branch at v_1 is not K_2. Suppose at least two branches are not K_2. Let B_1 and B_2 be such branches. Let v_i and v_j be the pendant vertices of B_1 and B_2 respectively. Let v_k be the neighbor of v_i. Then clearly, $V(G) \setminus \{v_i, v_j, v_k\}$ is a total resolving set with cardinality $n-3$, which is a contradiction. Thus exactly one branch at v_1 is not K_2. Let B_1 be such a branch. Then we claim that B_1 is P_3. Suppose that $|B_1| = t \geq 4$. Let $X = V(B_1) \setminus \{v_1\}$. Clearly, $V(G) \setminus X$ is a total resolving set with cardinality $n - (t-1)$, which is a contradiction. Hence $G \cong B_{1,t},\ t \geq 3$. Similarly, we can prove if G has exactly two exterior major vertices, then $G \cong B_{s,t},\ s,t \geq 2$.
Case 2 : One branch at v_1 contains a cycle.

Let B_1 be such a branch and $V(B_1) = \{v_1, v_2, \ldots, v_s\}$. We consider the following two subcases.
Subcase 2.1 : $|V(B_1)| = s \geq 4$.

We claim that B_1 is complete and others are K_2. Suppose B_1 is not complete. By Theorem 2.2, $tr(B_1) \leq s-2$. Then there exist $v_2, v_3 \in V(B_1)$ such that $V(B_1) \setminus \{v_2, v_3\}$ is a total resolving set of B_1. Let v_n be a pendant vertex of G in B_2. Then clearly, $V(G) \setminus \{v_2, v_3, v_n\}$ is a total resolving set of G with cardinality $n-3$, which is a contradiction. Therfore B_1 is complete. By Theorem 2.2, $tr(B_1) = n-1$. Then there exists a vertex $v_s \in V(B_1)$ such that $V(B_1) \setminus \{v_s\}$ is a total resolving set of B_1. Next, we claim that $B_2, B_3, \ldots, B_r$ are K_2. First, we claim that G contains exactly one exterior major vertex. If G contains at least two exterior major vertices v_j and v_k, then let v_j' and v_k' be the terminal vertices of v_j and v_k respectively. Clearly, $V(G) \setminus \{v_s, v_j', v_k'\}$ is a total resolving set of G with cardinality $n-3$, which is a contradiction. Hence G contains exactly one exterior major vertex.

Next, we claim that v_1 is the exterior major vertex. Suppose not. Clearly, $r = 2$ and B_2 contains exactly one exterior major vertex. Let v_n be the pendant vertex of B_2. Then clearly, $V(G) \setminus \{v_i, v_1, v_n\}$ is a total resolving set of G with cardinality $n-3$, which is a contradiction. Hence v_1 is the exterior major vertex. Now, we claim that each B_i is K_2. Suppose not. Let B_2 be such a branch and v_n be a pendant vertex of B_2 and v_{n-1} be its neighbor. Then clearly, $V(G) \setminus \{v_s, v_{n-1}, v_n\}$ is a total resolving set of G, which is a contradiction. Thus B_1 is complete and others are K_2. Hence $K_{1,s} * K_t$, $s \geq 1$ and $t \geq 3$.
Subcase 2.2 : $|V(B_1)| = 3$.

Then B_1 is K_3. Let $V(B_1) = \{v_1, v_2, v_3\}$. First, we claim that G contains exactly one exterior major vertex. Suppose G contains more than one exterior major vertex. Let v_i and v_j be such exterior major vertices and v_i' and v_j' be the terminal vertices of v_i and

v_j respectively. Then clearly, $V(G) \setminus \{v_3, v_i', v_j'\}$ is a total resolving set with cardinality $n-3$, which is a contradiction. Thus G contains exactly one exterior major vertex. If v_1 is the exterior major vertex, then we claim B_i is K_2 for all $2 \leq i \leq r$.

Suppose not. Without loss of generality, let $|B_2| \geq 3$. Let v_i be the pendant vertex of B_2 and v_j be the neighbor of v_i. Then $V(G) \setminus \{v_i, v_j, v_3\}$ is a total resolving set with cardinality $n-3$, which is a contradiction. Hence $G \cong K_{1,s} * K_3$. If v_1 is not the exterior major vertex, then clearly, $r = 2$ and neighbor of v_1 in B_2 is the exterior major vertex. Let v_i be the exterior major vertex. Then we claim $d(v_i, x) = 1$ for all $x \in V(B_2) \setminus \{v_i\}$. Suppose that $d(v_i, y) \geq 2$ for some pendant vertex $y \in V(B_2) \setminus \{v_i\}$. Let x be the neighbor of y. Then $V(G) \setminus \{v_3, x, y\}$ is a total resolving set with cardinality $n-3$, which is a contradiction. Hence $G \cong K_3\ e\ K_{1,s}$, $s \geq 3$.

Case 3 : Two branches at v_1 contain a cycle.

Let B_1, B_2 be such branches. First we claim that $r = 2$. Suppose $r \geq 3$. Let B_3 be a tree branch and v_i be a pendant vertex of B_3. Then there exist $v_j \in B_1$ and $v_k \in B_2 (j \neq k)$ such that $V(G) \setminus \{v_i, v_j, v_k\}$ is a total resolving set with cardinality $n-3$, which is a contradiction. Thus $r = 2$. Now, we claim that one branch is K_3 and another one is either complete or $K_1 + (K_2 \cup K_1)$. First we claim that either $|V(B_1)| = 3$ or $|V(B_2)| = 3$. Suppose $|V(B_1)| \geq 4$ and $|V(B_2)| \geq 4$. Let v_2, v_3 be the neighbors of v_1 and $v_2 \in V(B_1)$, $v_3 \in V(B_2)$. Then clearly, $V(G) \setminus \{v_1, v_2, v_3\}$ is a total resolving set of G with cardinality, $n-3$, which is a contradiction. Thus either $|V(B_1)| = 3$ or $|V(B_2)| = 3$. Without loss of generality, let $|V(B_1)| = 3$ and hence B_1 is K_3. Let $V(B_1) = \{v_1, v_2, v_3\}$ and $V(B_2) = \{v_1, v_4, v_5, \ldots, v_n\}$. Then $|V(B_2)| = n-2$. Next, we claim that B_2 is either complete or $K_1 + (K_2 \cup K_1)$. Suppose B_2 is neither complete nor $K_1 + (K_2 \cup K_1)$. Since B_2 is not complete, $tr(B_2) \leq n-4$. Let $V(B_2) \setminus \{v_{n-1}, v_n\}$ be a total resolving set of B_2. Clearly, $V(G) \setminus \{v_2, v_{n-1}, v_n\}$ is a total resolving set, which is a contradiction. Thus B_2 is either complete or $K_1 + (K_2 \cup K_1)$. Hence $K_3 * K_t$, $t \geq 3$ or $2K_3 + e$.

The converse can be easily verified. ∎

Notation 3.3. Let $\mathscr{G}$ be the collection of graphs G such that G is the union of two distinct paths $P_1 : x_1x_2 \ldots x_r$, $P_2 : y_1y_2 \ldots y_s$, $r \leq s$ and $x_1y_1 \in E(G)$, $x_iy_i \in E(G)$ for at least one i, $2 \leq i \leq r$.

Theorem 3.4. If G is a bipartite graph that is not a path, then $tr(G) = 2$ if and only if $G \in \mathscr{G}$.

Proof: Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $V(G) = S \cup T$.

Let $tr(G) = 2$ and $W = \{v_1, v_2\}$ be a total resolving set of G. Let $v_1 \in S$. Then $v_2 \in T$. Let d be the diameter of G and (x, y) be the representation of any vertex. Then $x \sim y \leq 1$. Since v_1 and v_2 are in distinct partite sets, $d(u, v_1) \neq d(u, v_2)$ for all $u \in V(G) \setminus \{v_1, v_2\}$ and hence $x \sim y = 1$. By Remark 2.3, $|N_l(W)| \leq 3$, for all $1 \leq l \leq d-1$. But the possible representations of the vertices of $N_l(W)$ are $(l, l+1)$ and $(l+1, l)$ for all $1 \leq l \leq d-1$. It follows that $|N_l(W)| \leq 2$ for all $1 \leq l \leq d-1$. Thus the possible representations of the vertices of $V(G) \setminus W$ are $(1,2), (2,3), \ldots, (d-1, d)$ and $(2,1), (3,2), \ldots, (d, d-1)$. Let $X = \{(1,2), (2,3), \ldots, (d-1, d)\}$ and $Y = \{(2,1), (3,2), \ldots, (d, d-1)\}$. We define $A = \{a \in V(G) \ / \ r(a|W) \in X\}$ and $B = \{b \in V(G) \ / \ r(b|W) \in Y\}$. Therefore $\langle A \cup \{v_1\} \rangle$ and $\langle B \cup \{v_2\} \rangle$ are paths in G. Then we can easily verify that $G \in \mathscr{G}$.

The converse can be easily verified. ∎

Open Problem 3.5. If G is a 2-connected graph of order $n \geq 4$, then characterize graphs for which $tr(G) = n - 2$.

Open Problem 3.6. Characterize non-bipartite graphs for which $tr(G) = 2$.

Acknowledgement

The research work of the second author is supported by the University Grants Commission, New Delhi through Basic Science Research Fellowship (vide Sanction No.F.7-201/2007(BSR).

References

[1] F. Buckely, and F. Harary, Distance in graphs, Addison Wesley, Reading MA, 1990.

[2] G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, Resolvability in graphs and metric dimension of a graph, Discrete Appl. Math. 105(2000), 99-113.

[3] G. Chartrand and P. Zhang, Introduction to Graph Theory, Tata McGraw Hill Education Private Limited, New Delhi(2006).

[4] F. Harary and R. A. Melter, On the metric dimension of a graph, Ars Combin. 2(1976), 191 - 195.

[5] J. Paulraj Joseph and N. Shunmugapriya, Total Resolving Number of a Graph, Indian Journal of Mathematics, Vol 57, No. 3(2015), 323-343.